FOR THE
IB DIPLOMA

Mathematics

APPLICATIONS AND INTERPRETATION HL

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WORKED SOLUTIONS

DYNAMIC LEARNING

HODDER EDUCATION

1 Exponents and logarithms

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 1A

14

$$\begin{aligned} 8^{-\frac{4}{3}} &= (2^3)^{-\frac{4}{3}} \\ &= 2^{-4} \\ &= \frac{1}{16} \end{aligned}$$

15

$$\begin{aligned} \left(\frac{1}{4}\right)^{-\frac{1}{2}} &= (2^{-2})^{-\frac{1}{2}} \\ &= 2^1 \\ &= 2 \end{aligned}$$

16

$$\begin{aligned} \left(\frac{4}{9}\right)^{\frac{3}{2}} &= \left(\frac{2}{3}\right)^3 \\ &= \frac{8}{27} \end{aligned}$$

17

$$\begin{aligned} \sqrt[3]{x^2} \times \sqrt[4]{x} &= x^{\frac{2}{3}} \times x^{\frac{1}{4}} \\ &= x^{\frac{2}{3}+\frac{1}{4}} \\ &= x^{\frac{11}{12}} \end{aligned}$$

18

$$\frac{3}{\sqrt[3]{x}} + 2\sqrt{x} = 3x^{-\frac{1}{3}} + 2x^{\frac{1}{2}}$$

19

$$\frac{1}{3\sqrt{x^3}} = \frac{1}{3}x^{-\frac{3}{2}}$$

20

$$x^{\frac{3}{2}} = 2^{-3}$$

$$x = 2^{-2} = \frac{1}{4}$$

21

$$x^{-\frac{1}{2}} = \frac{2}{5}$$

$$x^{\frac{1}{2}} = \frac{5}{2}$$

$$x = \frac{25}{4}$$

22

$$\frac{x\sqrt{x}}{\sqrt[3]{x}} = \frac{x^{\frac{3}{2}}}{x^{\frac{1}{3}}} = x^{\frac{3}{2}-\frac{1}{3}} = x^{\frac{7}{6}}$$

23

$$\frac{x}{x^2\sqrt{x}} = \frac{x^1}{x^{\frac{5}{2}}} = x^{1-\frac{5}{2}} = x^{-\frac{3}{2}}$$

24

$$\left(x \times \sqrt[3]{x}\right)^2 = \left(x^{\frac{4}{3}}\right)^2 = x^{\frac{8}{3}}$$

25

$$\left(\frac{1}{2\sqrt{x}}\right)^3 = \frac{1}{8}x^{-\frac{3}{2}}$$

26

$$\frac{x^2 + \sqrt{x}}{x\sqrt{x}} = x^{\frac{1}{2}} + x^{-1}$$

27

Numerator is difference of two squares:

$$\frac{\left(x+\sqrt{x}\right)\left(x-\sqrt{x}\right)}{\sqrt{x}} = \frac{x^2 - x}{\sqrt{x}} = x^{\frac{3}{2}} - x^{\frac{1}{2}}$$

28

$$\frac{1}{3x\sqrt{x}} = \frac{1}{3}x^{-\frac{3}{2}}$$

29

$$\frac{x^2 + 3\sqrt{x}}{2x} = \frac{1}{2}x^1 + \frac{3}{2}x^{-\frac{1}{2}}$$

30 $y^4 = \left(2\sqrt[3]{x^2}\right)^4 = \left(2x^{\frac{2}{3}}\right)^4 = 16x^{\frac{8}{3}}$

31 $\sqrt[3]{y} = \left(27x^{\frac{1}{2}}\right)^{\frac{1}{3}} = 3x^{\frac{1}{6}}$

32 $y = \left(\sqrt{x}\right)^3 = x^{\frac{3}{2}}$

33 $y^3 = \left(\frac{2}{3\sqrt{x}}\right)^3 = \frac{8}{27}x^{-\frac{3}{2}}$

34 $\left(\sqrt{3}\right)^x = 9^{x-1}$

$3^{\frac{x}{2}} = 3^{2(x-1)}$

$\frac{x}{2} = 2x - 2$

$\frac{3}{2}x = 2$

$x = \frac{4}{3}$

35 $\left(\sqrt{2}\right)^x = 4^{x+2}$

$2^{\frac{x}{2}} = 2^{2(x+2)}$

$\frac{x}{2} = 2x + 4$

$\frac{3}{2}x = -4$

$x = -\frac{8}{3}$

36 $\left(\sqrt[3]{2}\right)^{2x} = 8^{x+1}$

$2^{\frac{2x}{3}} = 2^{3(x+1)}$

$\frac{2x}{3} = 3x + 3$

$\frac{7}{3}x = -3$

$x = -\frac{9}{7}$

37 $\left(\sqrt[3]{3}\right)^{4x} = 9^{x-3}$

$3^{\frac{4x}{3}} = 3^{2(x-3)}$

$\frac{4x}{3} = 2x - 6$

$\frac{2}{3}x = 6$

$x = 9$

38

$$y = x^{\frac{3}{2}}$$

$$x = y^{\frac{2}{3}}$$

39

$$x^{\frac{2}{3}} = 3^2$$

$$|x| = 3^3 = 27$$

$$x = \pm 27$$

40

$$x^{\frac{1}{2}} = 2x^{\frac{1}{3}}$$

$$x^{\frac{1}{6}} = 2$$

$$x = 64$$

Exercise 1B

18 a $\log_{10}(ab^4) = \log_{10} a + 4\log_{10} b = x + 4y$

b $\log_{10}\left(\frac{a^2 b}{c^5}\right) = 2\log_{10} a + \log_{10} b - 5\log_{10} c = 2x + y - 5z$

c $\log_{10}(10a^2b^3) = \log_{10} 10 + 2\log_{10} a + 3\log_{10} b = 1 + 2x + 3y$

19 a $\log_{10}(100\sqrt{a}) = \log_{10} 100 + \frac{1}{2}\log_{10} a = 2 + \frac{1}{2}x$

b $\log_{10}\left(\frac{b}{10c^5}\right) = \log_{10} b - \log_{10} 10 - 5\log_{10} c = y - 1 - 5z$

20

$$2\ln a + 6\ln b = \ln(a^2b^6)$$

21

$$\frac{1}{3}\ln x - \frac{1}{2}\ln y = \ln\left(\frac{\sqrt[3]{x}}{\sqrt{y}}\right)$$

22

$$\log_{10}(x + 3) = 3$$

$$x + 3 = 1000$$

$$x = 997$$

23

$$\log_{10}(2x - 4) = 1$$

$$2x - 4 = 10$$

$$x = 7$$

24 **a** $5^x = 10$

$x\log_{10} 5 = \log_{10} 10 = 1$

$x = \frac{1}{\log_{10} 5} = 1.43$

b $2 \times 3^x + 6 = 20$

$3^x = 7$

$x\ln 3 = \ln 7$

$x = \frac{\ln 7}{\ln 3} = 1.77$

25 $3 \times 1.1^x = 20$

$x\ln 1.1 = \ln\left(\frac{20}{3}\right)$

$x = \frac{\ln\left(\frac{20}{3}\right)}{\ln 1.1} = 19.9$

26 **a** $\log_{10}\left(\frac{1}{\sqrt{10}}\right) = \log_{10}\left(10^{-\frac{1}{2}}\right) = -\frac{1}{2}$

b $\log_x 27 = -\frac{1}{3}$

$x = 27^{-\frac{1}{3}} = \frac{1}{3}$

27 $\log_x 32 = 5$

$x^5 = 32$

$x = 2$

28 $\log_x 64 = 3$

$x^3 = 64$

$x = 4$

29 **a** $5 \times 6^x = 12 \times 3^x$

$2^x = \frac{12}{5} = 2.4$

b $x = \frac{\ln(2.4)}{\ln 2} = 1.26$

30 $8^{3x+1} = 4^{x-3}$

$2^{3(3x+1)} = 2^{2(x-3)}$

$9x + 3 = 2x - 6$

$7x = -9$

$x = -\frac{9}{7}$

31 $5^{2x+3} = 9^{x-5} = 3^{2(x-5)}$

$(2x+3)\ln 5 = 2(x-5)\ln 3$

$x(2\ln 5 - 2\ln 3) = -10\ln 3 - 3\ln 5$

$x = \dfrac{10\ln 3 + 3\ln 5}{2\ln 3 - 2\ln 5}$

32 **a** $R(0) = 10$

b When $R = 5$, $0.9^t = 0.5$

$t = \dfrac{\ln 0.5}{\ln 0.9} = 6.58$

33 **a** **i** $B(0) = 1000$

ii $B(2) = 1210$

b $B(t) = 2000$ so $1.1^t = 2$

$t = \dfrac{\ln 2}{\ln 1.1} = 7.27$ hours

34 $P_1 = P_2$

$\left(\dfrac{1.2}{1.1}\right)^t = 2$

$t = \dfrac{\ln 2}{\ln\left(\frac{1.2}{1.1}\right)} = 7.97$ years

35

$\begin{cases} \log_{10} x + \log_{10} y = 3(1) \\ \log_{10} x - 2\log_{10} y = 0(2) \end{cases}$

$(1) - (2): 3\log_{10} y = 3$

$\log_{10} y = 1, \log_{10} x = 2$

$y = 10, x = 100$

36

$\begin{cases} \log_{10} x + \ln y = 1(1) \\ 2\log_{10} x + 3\ln y = 4(2) \end{cases}$

$3(1) - (2): \log_{10} x = -1$

$\ln y = 2$

$x = 0.1, y = e^2$

37 $2^{5-3x} = 3^{2x-1}$

$(5-3x)\ln 2 = (2x-1)\ln 3$

$(2\ln 3 + 3\ln 2)x = 5\ln 2 + \ln 3$

$x\ln(3^2 \times 2^3) = \ln(2^5 \times 3)$

$x\ln 72 = \ln 96$

$$x = \frac{\ln 96}{\ln 72}$$

38 $R = k \times 2^{\frac{t}{2}}$

$R(0) = k$ and when $R(t) = 10k$, $2^{\frac{t}{2}} = 10$

$t = 2\log_2 10 = 6.64$

39 **a** $\log_a(x^2) = b$

$x^2 = a^b$

$x = \pm a^{\frac{b}{2}}$

Then the product of possible values of x would be $\left(a^{\frac{b}{2}}\right) \times \left(-a^{\frac{b}{2}}\right) = -a^b$

b $(\log_a x)^2 = b$

$\log_a x = \pm\sqrt{b}$

$x = a^{\pm\sqrt{b}}$

Then the product of possible values of x would be $\left(a^{\sqrt{b}}\right) \times \left(a^{-\sqrt{b}}\right) = 1$

Exercise 1C

11 $u_1 = 3, r = \frac{1}{4}$

$$S_\infty = \frac{u_1}{1-r} = \frac{3}{\left(\frac{3}{4}\right)} = 4$$

12 $u_1 = 5, r = -\frac{1}{4}$

$$S_\infty = \frac{u_1}{1-r} = \frac{5}{\left(\frac{5}{4}\right)} = 4$$

13 $u_1 = 2, r = -\frac{1}{3}$

$$S_\infty = \frac{u_1}{1-r} = \frac{2}{\left(\frac{4}{3}\right)} = \frac{3}{2}$$

14 $u_1 = 8, S_\infty = \frac{u_1}{1-r} = 6$

$$1 - r = \frac{u_1}{S_\infty} = \frac{8}{6}$$

$$r = -\frac{1}{3}$$

15 $u_1 = 3, S_\infty = \frac{u_1}{1-r} = 4$

$$1 - r = \frac{u_1}{S_\infty} = \frac{3}{4}$$

$r = \frac{1}{4}$

16 $r = \frac{1}{3}, S_\infty = \frac{u_1}{1-r} = 3$

$u_1 = S_\infty(1-r) = 2$

$u_2 = u_1 r = \frac{2}{3}$

$u_3 = u_2 r = \frac{2}{9}$

17 $u_1 r = 2, S_\infty = \frac{u_1}{1-r} = 9$

$u_1 = 9(1-r)$

Substituting:

$9r(1-r) = 2$

$9r^2 - 9r + 2 = 0$

$(3r-1)(2r-2) = 0$

$r = \frac{1}{3} \text{ or } \frac{2}{3}$

[Check for convergence: Both solutions are valid since $|r| < 1$]

Tip: Even if you are not discarding solutions, it is worth explicitly observing that you have checked for solution validity. We know that S_∞ exists as a finite value and that for convergence $|r| < 1$, so would discard any solution values outside this interval.

18 Changing the dummy variable to k, so that r can be used as the common ratio of the series:

$$S_\infty = \sum_{k=0}^{\infty} \left(\frac{2}{5}\right)^k = \sum_{k=1}^{\infty} \left(\frac{2}{5}\right)^{k-1} = u_1 \sum_{k=1}^{\infty} r^{k-1}$$

$u_1 = 1, r = \frac{2}{5}$

a $u_1 = 1$

$u_2 = \frac{2}{5}$

$u_3 = \frac{4}{25}$

b

$$S_\infty = \frac{u_1}{1-r} = \frac{1}{\left(\frac{3}{5}\right)} = \frac{5}{3}$$

19 Changing the dummy variable to k, so that r can be used as the common ratio of the series:

$$\sum_{k=0}^{\infty} \frac{2}{3^k} = 2\sum_{k=1}^{\infty} \frac{1}{3^{k-1}} = u_1 \sum_{k=1}^{\infty} r^{k-1}$$

$u_1 = 2, r = \frac{1}{3}$

$|r| < 1$ so the series converges.

$$S_\infty = \frac{u_1}{1-r} = \frac{2}{1-\frac{1}{3}} = 3$$

20 a $u_1 = 3, r = -\frac{x}{9}$

Series converges for $|r| < 1$

$$\left|-\frac{x}{9}\right| < 1$$

$|x| < 9$

b $S_\infty = \frac{u_1}{1-r} = \frac{3}{1-\left(\frac{2}{9}\right)} = \frac{27}{7}$

21 a $u_1 = 5, r = (x-3)$

Series converges for $|r| < 1$

$|x-3| < 1$

$2 < x < 4$

b When the series converges,

$$S_\infty = \frac{u_1}{1-r} = \frac{5}{1-(x-3)} = \frac{5}{4-x}$$

22 a $u_1 = 2, r = 2x$

Series converges for $|r| < 1$

$|2x| < 1$

$$|x| < \frac{1}{2}$$

b When the series converges,

$$S_\infty = \frac{u_1}{1-r} = \frac{2}{1-2x}$$

23 $u_2 = u_1 r = -\frac{6}{5}, S_\infty = \frac{u_1}{1-r} = 5$

$u_1 = 5(1-r)$

Substituting: $5r(1-r) = -\dfrac{6}{5}$

$25r^2 - 25r - 6 = 0$

$(5r-6)(5r+1) = 0$

$$r = \frac{6}{5} \text{ or } -\frac{1}{5}$$

The series converges to a finite S_∞ so $|r| < 1$. Therefore $r = -\frac{1}{5}$

$u_1 = 5(1-r) = 6$

24 a $u_1 = x, r = 4x^2$

Series converges for $|r| < 1$

$4x^2 < 1$

$2x < 1$ ($|2x| = 2x$ since x is known to be positive)

$0 < x < \frac{1}{2}$

b

$$S_\infty = \frac{u_1}{1-r} = \frac{x}{1-4x^2}$$

25 a Changing the dummy variable to k, so that r can be used as the common ratio of the series:

$$\sum_{k=0}^{\infty} \frac{x^{k+1}}{2^k} = x\sum_{k=1}^{\infty} \frac{x^{k-1}}{2^{k-1}} = x\sum_{k=1}^{\infty} \left(\frac{x}{2}\right)^{k-1} = u_1 \sum_{k=1}^{\infty} r^{k-1} = 3$$

$u_1 = x, r = \frac{x}{2}$

Series converges for $|r| < 1$ so $\left|\frac{x}{2}\right| < 1; |x| < 2$

$$S_\infty = \frac{u_1}{1-r} = \frac{x}{1-\left(\frac{x}{2}\right)} = 3$$

$x = 3 - \frac{3}{2}x$

$\frac{5}{2}x = 3$

$x = \frac{6}{5}$

This is within the region of convergence, so is a valid solution.

b $|x| < 2$

26

$$S_\infty = \frac{u_1}{1-r} = 27(1)$$

$u_1 + u_1 r + u_1 r^2 = 19(2)$

(1): $u_1 = 27(1-r)$

Substituting into (2):

$27(1-r)(1+r+r^2) = 19$

$27(1-r^3) = 19$

$r^3 = \frac{8}{27}$

$r = \frac{2}{3}$

Then (1): $u_1 = 27(1-r) = 9$

Worked solutions

Exercise 1D

9 log-log graph appears straight, passing through approximately $(0,0.7)$ and $(1,4.7)$

Gradient $= 4$

$\log y = 4\log x + 0.7$

$y = 10^{0.7}x^4 \approx 5x^4$

10 log-log graph appears straight, passing through approximately $(0,1)$ and $(1,-1)$

Gradient $= -2$

$\log y = -2\log x + 1$

$y = 10x^{-2}$

11 Semi-log graph appears straight, passing through approximately $(1,2.1)$ and $(8,6.9)$

Gradient $\approx \frac{4.8}{7}$

$\ln y - 2.1 = \frac{4.8}{7}(x-1)$

$\ln y = \frac{9.9}{7} + \frac{4.8}{7}x$

$y \approx \mathrm{e}^{\frac{9.9}{7}} \times \mathrm{e}^{\frac{4.8}{7}x} \approx 4 \times 2^x$

12 Semi-log graph appears straight, passing through approximately $(1,2.5)$ and $(5,-3)$

Gradient $\approx -\frac{5.5}{4}$

$\ln y + 3 = -\frac{5.5}{4}(x-5)$

$\ln y = \frac{15.5}{4} - \frac{5.5}{4}x$

$y \approx \mathrm{e}^{\frac{15.5}{4}} \times \mathrm{e}^{-\frac{5.5}{4}x} \approx 50 \times \left(\frac{1}{4}\right)^x$

13 Semi-log graph appears approximately straight, with best fit line passing through approximately $(2,4)$ and $(15,25)$

Gradient $\approx \frac{21}{13}$

$\ln y - 4 = \frac{21}{13}(x-2)$

$\ln y = \frac{10}{13} + \frac{21}{13}x$

$y \approx \mathrm{e}^{\frac{10}{13}} \times \mathrm{e}^{\frac{21}{13}x} \approx 2 \times 5^x$

14 **a** $\ln X = -kt + \ln X_0$

b

t	1	2	3	4
X	12	7	4	3
$\ln X$	2.48	1.95	1.39	1.10

From calculator linear regression on these four data points: $\ln X = 2.91 - 0.472t$

Then $X_0 = e^{2.905} = 18.3$ and $k = 0.472$

c When $X = 0.5X_0$, $e^{-kt} = 0.5$

$$t = -\frac{1}{k}\ln 0.5 = 1.47 \text{ minutes}$$

15 a $\log F = n\log d + \log k$

Tip: Using natural log instead of base 10 will yield the same answer in part **b.**

b

d	1	2	3	4
F	0.15	0.0375	0.0167	0.00938
$\log F$	−0.824	−1.43	−1.78	−2.03
$\log d$	0	0.301	0.477	0.602

From calculator linear regression on these four data points: $\log F = -2.0\log d - 0.824$

Then $k = 10^{-0.824} = 1.5$ and $n = -2.0$

16 a $\ln m = y\ln b + \ln a$

y	1	2	3	4	5	6	7	8
m	159	192	256	297	668	398	1369	1656
$\ln m$	0	0.69	1.1	1.39	1.61	1.79	1.95	2.08

From calculator linear regression on $\ln m$ against y: $\ln m \approx 0.337y + 4.57$

So $b = e^{0.337} \approx 1.40$ and $a = e^{4.57} = 96.3$

b The average percentage increase in miles added is therefore 40% per year.

c Extrapolating, under the assumption that the percentage increase will remain consistent year on year, when $y = 18$, m is projected to be $96.3 \times 1.40^{18} \approx 42\,000$ miles

17 a $y = ad^n$

$\ln y = n\ln d + \ln a$

d	58	108	150	228	779	1434
y	88	225	365	687	4380	10585
$\ln d$	4.06	4.68	5.01	5.43	6.66	7.27
$\ln y$	4.48	5.42	5.9	6.53	8.38	9.27

From calculator linear regression on $\ln y$ against $\ln d$: $\ln y \approx 1.50\ln d - 1.59$

So $n = 1.5$ and $a = e^{-1.59} = 0.2$

b For $d = 2871$, the model would project $y = 0.2 \times 2871^{1.5} \approx 30767$ days ≈ 84 years

18 $F = kx^{-n}$ so $\ln F = \ln k - n\ln x$

x	1	2	3	4
F	89	28	10	6
$\ln x$	0	0.69	1.1	1.39
$\ln F$	4.49	3.33	2.3	1.79

From calculator linear regression on F against $\ln x$: $\ln F \approx -1.98\ln x + 4.55$

So $n = 2$ and $k = e^{4.55} = 94.6$

19 a

R	1	2	3	4	5	6
P	6.5	3.4	2.1	1.6	1.2	1.1
$\ln R$	0	0.69	1.1	1.39	1.61	1.79
$\ln P$	1.87	1.22	0.74	0.47	0.18	0.1

i PMRCC for $\ln P$ against $\ln R$ is $r = -0.9983$

ii PMRCC for $\ln P$ against R is $r = -0.965$

b The stronger linear correlation is from the log-log graph, with line equation $\ln P = k + n \ln R$, so model as $P = aR^n$

c From calculator linear regression on $\ln P$ against $\ln R$: $\ln P \approx -1.02 \ln R + 1.89$

So $n = -1.0$ and $a = \mathrm{e}^{1.89} = 6.6$

20 a

R	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
F	4355	2872	1748	1745	1665	1520	1135	968	901	835	824	738	706	651	623	618
$\ln R$	0	0.69	1.1	1.39	1.61	1.79	1.95	2.08	2.2	2.3	2.4	2.48	2.56	2.64	2.71	2.77
$\ln F$	8.38	7.96	7.47	7.46	7.42	7.33	7.03	6.88	6.8	6.73	6.71	6.6	6.56	6.48	6.43	6.43

i PMRCC for $\ln F$ against $\ln R$ is $r = -0.990$

ii PMRCC for $\ln F$ against R is $r = -0.945$

b The stronger linear correlation is from the log-log graph, with line equation $\ln F = k + n \ln R$, so model as $F = aR^n$

c From calculator linear regression on $\ln F$ against $\ln R$: $\ln F \approx -0.724 \ln R + 8.43$

So $n = -0.72$ and $a = \mathrm{e}^{8.43} = 4600$

d When $F = 10$, the model projects $R = \left(\frac{F}{a}\right)^{\frac{1}{n}} \approx 4990$

21 $y = ax^n$

Since x takes negative values, use logarithm of absolute value to transform:

$\ln|y| = \ln|a| + n \ln|x|$

For the model to make sense with real values predicted, n must be an even integer.

From calculator linear regression on $\ln|y|$ against $\ln|x|$: $|y| \approx 2.0 \ln x + 1.58$

So $n = 2$ and $|a| = \mathrm{e}^{1.58} = 4.88$

Since $y < 0$ at all data points, $a = -4.88$

22 a $T = A\mathrm{e}^{-kt} + c$ where c is the background temperature.

Then $(T - c) = A\mathrm{e}^{-kt}$

$\ln(T - c) = -kt + \ln A$

t	1		2	3	4
$T - 20$	7		4.5	3	2
$\ln(T - 20)$	1.95		1.5	1.1	0.69

From calculator linear regression on $\ln(T - 20)$ against t:
$\ln(T - 20) \approx -0.416t + 2.35$

So $k = 0.416$ and $A = \mathrm{e}^{2.35} = 10.5$

b In this model, when $T = 37$, $Ae^{-kt} = 17$

$$t = -\frac{1}{k}\ln\left(\frac{17}{A}\right) = -1.15$$

Time of death was 1.15 hours before noon, or approximately 10: 50 am.

23 a As t gets large, the denominator tends to 1, so P approaches 10.

b

$$P = \frac{10}{1 + Je^{-\beta t}}$$

$$\frac{10}{P} = 1 + Je^{-\beta t}$$

$$\frac{10}{P} - 1 = Je^{-\beta t}$$

$$\ln\left(\frac{10}{P} - 1\right) = \ln J - \beta t$$

c

t	2	3	4	5
$\frac{10}{P} - 1$	0.667	0.538	0.449	0.370
$\ln\left(\frac{10}{P} - 1\right)$	-0.41	-0.62	-0.8	-0.99

From calculator linear regression on $\ln\left(\frac{10}{P} - 1\right)$ against t:

$$\ln\left(\frac{10}{P} - 1\right) \approx -0.195t - 0.0228$$

So $\beta = 0.195$ and $J = e^{-0.0228} = 0.977$

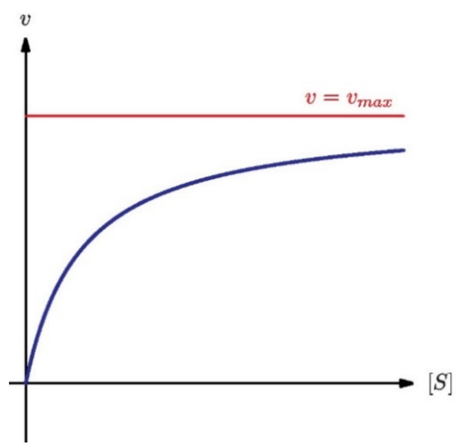
d When $P = 9$, the model predicts $t = \frac{1}{\beta}\left(\ln J - \ln\left(\frac{10}{9} - 1\right)\right) = 11.2$ years

e When $t = 0$, the model predicts $P = \frac{10}{1+J} = 5060$ rabbits

f Both calculations involve extrapolating beyond the observed data. The model has a very high correlation to the data ($r = -0.9995$) but this does not mean that extrapolation is reliable.

24 a If $[S] = K_m$ then $v = \frac{v_{max}}{2}$

b



c

$$\frac{1}{v} = \frac{K_m}{v_{max}[S]} + \frac{1}{v_{max}} = \frac{K_m}{v_{max}} \times \frac{1}{[S]} + \frac{1}{v_{max}}$$

The line would have gradient $\frac{K_m}{v_{max}}$ and y-intercept $\frac{1}{v_{max}}$

d i

$[S]$	1	2	3	4
v	2	3	3.5	3.7
$\frac{1}{[S]}$	1	0.5	0.33	0.25
$\frac{1}{v}$	0.5	0.33	0.29	0.27

From calculator linear regression $\frac{1}{v}$ against $\frac{1}{[S]}$: $\frac{1}{v} \approx 0.313\frac{1}{[S]} + 0.185$

ii $v_{max} = \frac{1}{0.185} = 5.42, K_m = 0.313 \times v_{max} = 1.70$

e i $v = \frac{v_{max}(K_m+[S]-K_m)}{K_m+[S]} = \frac{v_{max}(K_m+[S])}{K_m+[S]} - \frac{K_m}{K_m+[S]} = v_{max} - K_m\frac{v}{[S]}$

Gradient is K_m and y-intercept is v_{max}

ii

$[S]$	1	2	3	4
v	2	3	3.5	3.7
$\frac{v}{[S]}$	2	1.5	1.17	0.93

From calculator linear regression v against $\frac{v}{[S]}$: $v \approx 5.31 - 1.61\frac{v}{[S]}$

$v_{max} = 5.31, K_m = 1.61$

Mixed Practice

1 a

$$\left(\frac{4}{9}\right)^{-\frac{3}{2}} = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \left(\frac{3^2}{2^2}\right)^{\frac{3}{2}} = \left(\frac{3^3}{2^3}\right) = \frac{27}{8}$$

b $\log_{10}\left(\frac{1}{1000}\right) = \log_{10}(10^{-3}) = -3$

2 $\ln 4 + 2\ln 3 = \ln 4 + \ln 9 = \ln 36$

3 $2 \times 3^{x-2} = 54$

$3^{x-2} = 27 = 3^3$

$x - 2 = 3$

$x = 5$

4 $1.05^x = 2$

$x = \log_{1.05} 2 = 14.2$

5 $100^{x+1} = 10^{3x}$

$10^{2(x+1)} = 10^{3x}$

$2x + 2 = 3x$

$x = 2$

6 $\log_{10}(5x + 10) = 2$

$5x + 10 = 10^2 = 100$

$5x = 90$

$x = 18$

7 $3\ln x + 2 = 2(\ln x - 1)$

$\ln x = -4$

$x = e^{-4}$

8 **a** $\log_{10}(x^2y) = 2\log_{10} x + \log_{10} y = 2a + b$

b $\log_{10}\left(\frac{x}{yz^3}\right) = \log_{10} x - \log_{10} y - 3\log_{10} z = a - b - 3c$

c $\log_{10}\left(\sqrt{zx^3}\right) = \frac{1}{2}\log_{10} z + \frac{3}{2}\log_{10} x = \frac{1}{2}c + \frac{3}{2}a$

9 $u_1 = \frac{1}{3}, r = \frac{1}{3}$

$$S_\infty = \frac{u_1}{1-r} = \frac{\frac{1}{3}}{\left(\frac{2}{3}\right)} = \frac{1}{2}$$

10 $u_1 = 7, r = -\frac{5}{9}$

$$S_\infty = \frac{u_1}{1-r} = \frac{7}{\left(\frac{14}{9}\right)} = \frac{9}{2}$$

11 $r = \frac{3}{4}$

$$S_\infty = \frac{u_1}{1-r} = 12$$

$$u_1 = 12 \times \frac{1}{4} = 3$$

12 The semi-log plot shows a curve but the log-log plot appears to show a straight line, passing through $(1, -2)$ and $(2.6, -8)$

Gradient is then approximately $-\frac{6}{1.6} = -3.75$ and y-intercept is approximately 1.75.

$\ln y \approx 1.75 - 3.75\ln x$

$y \approx e^{1.75} \times x^{-3.75} \approx 6x^{-4}$

13 a $\log_{10}\left(\frac{a}{\sqrt{b}}\right) = \log_{10} a - \frac{1}{2}\log_{10} b = x - \frac{1}{2}y$

b

$$\log_{10}\left(\frac{a^2}{1000b^3}\right) = 2\log_{10} a - \log_{10} 1000 - 3\log_{10} b$$
$$= 2x - 3 - 3y$$

14 a $\ln 10 = \ln(2 \times 5) = \ln 2 + \ln 5 = x + y$

b $\ln 50 = \ln(2 \times 5^2) = \ln 2 + 2\ln 5 = x + 2y$

c $\ln 0.08 = \ln\left(\frac{2}{5^2}\right) = \ln 2 - 2\ln 5 = x - 2y$

15

$3 + 2\log 5 - 2\log 2 = \log 1000 + \log 25 - \log 4$

$= \log\left(\dfrac{1000 \times 25}{4}\right) = \log 6250$

16 $3\ln x + \ln 8 = 5$

$3(\ln x + \ln 2) = 5$

$\ln 2x = \dfrac{5}{3}$

$x = \dfrac{1}{2}e^{\frac{5}{3}}$

17 $4^{3x+5} = 8^{x-1}$

$2^{2(3x+5)} = 2^{3(x-1)}$

$6x + 10 = 3x - 3$

$3x = -13$

$x = -\dfrac{13}{3}$

18

$$\begin{cases} \log_{10} x + \log_{10} y = 5(1) \\ \log_{10} x - 2\log_{10} y = -1(2) \end{cases}$$

$(1) - (2): 3\log_{10} y = 6$

$\log_{10} y = 2, \log_{10} x = 3$

$x = 1000, y = 100$

19 $\log_x 8 = 6$

$x^6 = 8 = 2^3$

$x = 2^{\frac{1}{2}} = \sqrt{2}$

20 $3^{2x} = 2e^x$

$e^{2x\ln 3} = e^{x + \ln 2}$

$2x\ln 3 = x + \ln 2$

$x(2\ln 3 - 1) = \ln 2$

$x = \dfrac{\ln 2}{2\ln 3 - 1}$

21 $5^{2x+1} = 7^{x-3}$

$(2x+1)\ln 5 = (x-3)\ln 7$

$x(2\ln 5 - \ln 7) = -3\ln 7 - \ln 5$

$x = \dfrac{3\ln 7 + \ln 5}{\ln 7 - 2\ln 5}$

22 $12^{2x} = 4 \times 3^{x+1}$

$12^{2x} = 12 \times 3^x$

$144^x = 12 \times 3^x$

$\left(\dfrac{144}{3}\right)^x = 12$

$48^x = 12$

$x\ln 48 = \ln 12$

$x = \dfrac{\ln 12}{\ln 48}$

23 **a** $N(0) = 150$

b $N(3) = 3\,397$

c $N(t) = 1000$

$150\mathrm{e}^{1.04t} = 1000$

$t = \dfrac{1}{1.04}\ln\left(\dfrac{1000}{150}\right) = 1.82 \text{ hours}$

24 $\mathrm{e}^{1+3\ln x} = \mathrm{e} \times \mathrm{e}^{\ln x^3} = x^3\mathrm{e}$

25

$$\begin{aligned}\log_{10}\left(\frac{100^x}{10^y}\right) &= \log_{10}(100^x) - \log_{10}(10^y)\\ &= \log_{10}(10^{2x}) - \log_{10}(10^y)\\ &= 2x - y\end{aligned}$$

So, $p = 2$, $q = -1$

26 **a** $\log_{10} p^2 = 2\log_{10} p = 3$

b $\log_{10}\left(\frac{p}{q}\right) = \log_{10} p - \log_{10} q = 1.5 - 2.5 = -1$

c $\log_{10}(10q) = \log_{10} 10 + \log_{10} q = 1 + 2.5 = 3.5$

27 $u_1 = 15, r = 1.2$

$u_n = u_1 r^{n-1} = 231$

$r^{n-1} = \dfrac{231}{u_1}$

$$n - 1 = \frac{\ln\left(\frac{231}{u_1}\right)}{\ln r} \approx 15.0$$

$n = 16$

28 a $u_1 = 2 - 3x, r = 2 - 3x$

Series converges for $|r| < 1$

$|2 - 3x| < 1$

$\left|\frac{2}{3} - x\right| < \frac{1}{3}$

$\frac{1}{3} < x < 1$

b $S_\infty = \frac{u_1}{1-r} = \frac{1}{2}$

$\frac{2 - 3x}{3x - 1} = \frac{1}{2}$

$4 - 6x = 3x - 1$

$5 = 9x$

$x = \frac{5}{9}$

c If $S_\infty = -\frac{2}{3}$:

$\frac{2 - 3x}{3x - 1} = -\frac{2}{3}$

$6 - 9x = 2 - 6x$

$4 = 3x$

$x = \frac{4}{3}$

But $x = \frac{4}{3}$ is outside the interval of convergence from part **a** so S_∞ cannot equal $-\frac{2}{3}$

29 $S_\infty = \frac{u_1}{1-r} = 3u_1$

$1 - r = \frac{1}{3}$

$r = \frac{2}{3}$

30 a

y	1	4	7	11	14
C	3.0	2.7	2.5	2.2	2.0
$\ln y$	0	1.39	1.95	7.61	7.61
$\ln C$	1.1	0.99	0.92	0.79	0.69

i For the semi-log graph, $r = -0.999$

ii For the log-log graph, $r = -0.939$

b The greater correlation is the semi-log, so the best model is

$\ln C = \beta y + \alpha$

$C = e^{\alpha} e^{\beta y} = ab^y$ where $a = e^{\alpha}$ and $b = e^{\beta}$

c From calculator linear regression on $\ln C$ against y: $\ln C \approx 1.14 - 0.031y$

So $b = e^{-0.031} \approx 0.97$ and $a = e^{1.14} = 3.1$

d $C = 3.1e^{-0.031y}$

$$\int_0^{\infty} 3.1e^{-0.031y}\,dy = \left[-\frac{3.1}{0.031}e^{-0.031y}\right]_0^{\infty} = \frac{3.1}{0.031} = 101$$

The total amount of carbon dioxide absorbed from $y = 0$ according to the model is approximately 100 billion tonnes

31 $u_1(1 + r + r^2) = 62.755(1)$

$$S_{\infty} = \frac{u_1}{1 - r} = 440(2)$$

Since the series converges, $|r| < 1$

$(2): u_1 = 440(1 - r)$

Substituting into (1):

$440(1 - r^3) = 62.755$

$$1 - r^3 = \frac{62.755}{440}$$

$$r^3 = 1 - \left(\frac{62.755}{440}\right)$$

$$r = \frac{19}{20}$$

32

$$6^{3x} = 8^{x-1} = \frac{2^{3x}}{2^3}$$

$$6^x = \frac{2^x}{2}$$

$$3^x = \frac{1}{2}$$

$x \ln 3 = -\ln 2$

$$x = -\frac{\ln 2}{\ln 3}$$

33

$3^{x+1} = 3^x + 18$

$3(3^x) = 3^x + 18$

$2(3^x) = 18$

$3^x = 9$

$x = 2$

34 a $\ln k = \ln A - \frac{E_a}{RT} = \ln A - \frac{E_a}{R} \times \frac{1}{T}$

b

T	280	290	300	310	320
k	4.9×10^{-10}	2.1×10^{-9}	8.5×10^{-9}	3.1×10^{-8}	1×10^{-7}
$\frac{1}{T}$	0.0036	0.0034	0.0033	0.0032	0.0031
$\ln k$	-21.44	-19.98	-18.58	-17.29	-16.11

From the calculator, linear regression on $\ln k$ against $\frac{1}{T}$: $\ln k \approx 21.2 - 8.37 \times 10^{-5} \times \frac{1}{T}$

So $\frac{E_a}{R} = 11952$

$E_a = R \times (11952) = 99.2\text{kJ mol}^{-1}$

35

$$\begin{aligned}\ln x + \ln x^2 + \ln x^3 + \cdots + \ln x^{20} &= \ln x\,(1 + 2 + 3 + \cdots + 20)\\ &= 210 \ln x\end{aligned}$$

36
$$\begin{aligned}\log_3\left(\frac{1}{3}\right) + \log_3\left(\frac{3}{5}\right) + \log_3\left(\frac{5}{7}\right) + \cdots + \log_3\left(\frac{79}{81}\right) &= \log_3\left(\frac{1}{3} \times \frac{3}{5} \times \frac{5}{7} \times \ldots \times \frac{79}{81}\right)\\ &= \log_3\left(\frac{1}{81}\right)\\ &= -4\end{aligned}$$

37

$$\sum_{r=0}^{\infty} \frac{3^n + 4^n}{5^n} = \sum_{r=0}^{\infty}\left(\frac{3}{5}\right)^n + \sum_{r=0}^{\infty}\left(\frac{4}{5}\right)^n$$

This is the sum of two geometric series, both with first term 1, one with geometric ratio $r_1 = \frac{3}{5}$ and the other $r_2 = \frac{4}{5}$

$$\sum_{r=0}^{\infty} \frac{3^n + 4^n}{5^n} = \frac{1}{1 - \frac{3}{5}} + \frac{1}{1 - \frac{4}{5}} = \frac{5}{2} + 5 = \frac{15}{2}$$

38 a Common ratio is $r = \mathrm{e}^{-x}$. So $0 < r < 1$ for all positive x, and therefore the series will converge for all positive values of x.

b For positive x,

$$S_\infty = \frac{u_1}{1 - r} = \frac{\mathrm{e}^{-x}}{1 - \mathrm{e}^{-x}} = \frac{1}{\mathrm{e}^x - 1}$$

c $2 = \frac{1}{\mathrm{e}^x - 1}$

$$\mathrm{e}^x - 1 = \frac{1}{2}$$

$$x = \ln\left(\frac{3}{2}\right)$$

Worked solutions

39 $S_\infty = 27 = u_1(1 + r + r^2 + \cdots)$

But $u_1(r^3 + r^4 + r^5 + \cdots) = 1 = r^3 \times S_\infty$

So $r^3 = \frac{1}{27}$

$$r = \frac{1}{3}$$

40 a i $u_n = a + (n-1)d$

Then

$\frac{v_{n+1}}{v_n} = \frac{2^{a+nd}}{2^{a+(n-1)d}} = 2^d$ which is constant

ii $v_1 = 2^{u_1} = 2^a$

iii $v_n = 2^{a+d(n-1)}$

b i From part **ai**, $\{v_n\}$ is a geometric sequence with constant ratio $r = 2^d$

$S_n = \frac{v_1(r^n-1)}{r-1} = \frac{2^a(2^{dn}-1)}{2^d-1}$

ii Convergence occurs for $|r| < 1$

$|2^d| < 1$

$d < 0$

iii $S_\infty = \frac{v_1}{1-r} = \frac{2^a}{1-2^d}$

iv $\frac{2^a}{1-2^d} = 2^{a+1}$

$1 - 2^d = \frac{1}{2}$

$d = -1$

c $w_n = pq^{n-1}$

$z_n = \ln w_n = \ln p + (n-1)\ln q$

$$\sum_{i=1}^{n} z_i = n\ln p + \ln q \sum_{i=1}^{n}(i-1)$$

$$= n\ln p + \frac{n(n-1)}{2}\ln q$$

$$= \ln p^n + \ln q^{\frac{n(n-1)}{2}}$$

$$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right) = \ln k$$

$$k = p^n q^{\frac{n(n-1)}{2}}$$

2 Vectors

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 2A

34 $12 - 5 + x = 1 \Rightarrow x = -6$

$y + 2 + 9 = -2 \Rightarrow y = -13$

$-3 + z + 4 = 5 \Rightarrow z = 4$

35 a $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE} = \mathbf{a} + \frac{1}{2}\mathbf{b}$

b $\overrightarrow{EF} = \overrightarrow{EC} + \overrightarrow{CF} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

c $\overrightarrow{DG} = \overrightarrow{DA} + \overrightarrow{AG} = -\mathbf{b} + 2\mathbf{a}$

36 a $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = \mathbf{a} - \mathbf{b}$

b $\overrightarrow{ON} = \overrightarrow{OB} + \frac{1}{3}\overrightarrow{BA} = \frac{1}{3}(\mathbf{a} + 2\mathbf{b})$

c $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON} = -\frac{1}{2}\mathbf{a} + \frac{1}{3}(\mathbf{a} + 2\mathbf{b}) = \frac{1}{6}(4\mathbf{b} - \mathbf{a})$

37 a $3\mathbf{a} - \mathbf{c} + 5\mathbf{b} = 3\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -19 \end{pmatrix}$

b $|\mathbf{b} - 2\mathbf{a}| = \left|\begin{pmatrix} -4 \\ 3 \\ -11 \end{pmatrix}\right| = \sqrt{16 + 9 + 121} = \sqrt{146}$

38 $\left|\begin{pmatrix} 3k \\ -k \\ k \end{pmatrix}\right| = |k|\sqrt{9 + 1 + 1} = \sqrt{11}|k| = 22$

$|k| = 2\sqrt{11}$

$k = \pm 2\sqrt{11}$

39 $\left|\begin{pmatrix} 2 \\ 3t \\ t - 1 \end{pmatrix}\right| = 3 = \sqrt{4 + 9t^2 + (t - 1)^2}$

$4 + 9t^2 + t^2 - 2t + 1 = 9$

$10t^2 - 2t - 4 = 0$

$t = \dfrac{1 \pm \sqrt{41}}{10}$

40 $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \begin{pmatrix} 8 \\ -2 \\ 10 \end{pmatrix}$ N

The magnitude of the resultant force is $\sqrt{8^2 + (-2)^2 + 10^2} = \sqrt{168} = 2\sqrt{42}$ N.

41 $\mathbf{x} = \frac{1}{4}(\mathbf{b} - 3\mathbf{a}) = \frac{1}{4}\left(\begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right) = \frac{1}{4}\begin{pmatrix} 8 \\ 0 \\ -3 \end{pmatrix}$

42 $t\mathbf{b} = t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$t = -2$

43 a

$$\frac{1}{\sqrt{36 + 36 + 9}} = \frac{1}{9}$$

Unit vector is $\frac{1}{9}\begin{pmatrix} 6 \\ 6 \\ -3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

b $\left|\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}\right| = \sqrt{16 + 1 + 8} = 5$

A vector of magnitude 10 parallel to the vector is $\begin{pmatrix} 8 \\ -2 \\ 4\sqrt{2} \end{pmatrix}$.

44 $\mathbf{a} + p\mathbf{b} = \begin{pmatrix} 2 + 3p \\ p \\ 2 + 3p \end{pmatrix} = k\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$

Require $p = 2k$ and $2 + 3p = 3k$

Solving by substitution: $2 + 6k = 3k$

$k = -\frac{2}{3}$

Then $p = -\frac{4}{3}$

45 $\lambda\mathbf{x} + \mathbf{y} = \begin{pmatrix} 2\lambda + 4 \\ 3\lambda + 1 \\ \lambda + 2 \end{pmatrix} = k\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Require $\lambda + 2 = 0$

$\lambda = -2$

46

$\left|\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}\right| = \sqrt{16 + 1 + 1} = 3\sqrt{2}$

Then $\sqrt{2}\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ has magnitude 6.

47 $|\mathbf{a}+\lambda\mathbf{b}| = \left|\begin{pmatrix}2\lambda-2\\-\lambda\\2\lambda-1\end{pmatrix}\right| = \sqrt{4\lambda^2-8\lambda+4+\lambda^2+4\lambda^2-4\lambda+1} = 5\sqrt{2}$

$9\lambda^2-12\lambda+5=50$

$9\lambda^2-12\lambda-45=0$

$3\lambda^2-4\lambda-15=0$

$(3\lambda+5)(\lambda-3)=0$

$\lambda=3 \text{ or } -\frac{5}{3}$

48 $\overrightarrow{AB} = \begin{pmatrix}2+2t\\4+t\\-9-5t\end{pmatrix}$ so $AB = \sqrt{4+8t+4t^2+16+8t+t^2+81+90t+25t^2} = 3$

Squaring: $30t^2+106t+101=9$

$15t^2+53t+46=0$

$(t+2)(15t+23)=0$

$t=-2 \text{ or } -\frac{23}{15}$

49 a $\mathbf{m} = \frac{1}{2}(\mathbf{p}+\mathbf{q}) = \frac{1}{2}(3\mathbf{i}+3\mathbf{j}-4\mathbf{k})$

b Require $\overrightarrow{QR} = \overrightarrow{MQ} = \frac{1}{2}(-\mathbf{i}+5\mathbf{j}+2\mathbf{k})$

R has coordinates $\left(\frac{1}{2},\frac{13}{2},0\right)$

50

$$\begin{aligned}\left|\begin{pmatrix}2t\\t+3\\-2t-1\end{pmatrix}\right| &= \sqrt{4t^2+t^2+6t+9+4t^2+4t+1}\\ &= \sqrt{9t^2+10t+10}\\ &= \sqrt{\left(3t+\frac{5}{3}\right)^2+\frac{65}{9}}\end{aligned}$$

From the completed square form, this has minimum value $\frac{\sqrt{65}}{3}$.

Exercise 2B

15 a $\mathbf{a} = \begin{pmatrix}3\\-1\\5\end{pmatrix}, \mathbf{b} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

Line direction vector $\mathbf{d} = \mathbf{b}-\mathbf{a} = \begin{pmatrix}-4\\2\\-3\end{pmatrix}$

Line has equation $\mathbf{r} = \begin{pmatrix}3\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}-4\\2\\-3\end{pmatrix}$

b If $(0, 1, 5)$ lies on the line then $\begin{pmatrix}0\\1\\5\end{pmatrix} = \begin{pmatrix}3\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}-4\\2\\-3\end{pmatrix}$ for some λ

From the y element: $1 = 2\lambda - 1$ so $\lambda = 1$

This does not provide a consistent solution for the other two elements.

$(0, 1, 5)$ does not lie on the line.

16 The required line has vector equation $\mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\-3\end{pmatrix}$

$$\begin{cases} x = -1 + 2\lambda \\ y = 1 - \lambda \\ z = 2 - 3\lambda \end{cases}$$

17 Solving for λ:

$$\lambda = \frac{x+1}{2} = \frac{4-y}{3} = \frac{2z}{3}$$

Substituting $x = 3, y = -2, z = 2$:

$$\begin{cases} \dfrac{x+1}{2} = 2 \\ \dfrac{4-y}{3} = 2 \\ \dfrac{2z}{3} = \dfrac{4}{3} \neq 2 \end{cases}$$

The point $(3, -2, 2)$ does not lie on the given line.

18 **a**

$$\frac{(3-x)}{2} = \frac{(3z+1)}{4} = \lambda, y = -1$$

$$x = 3 - 2\lambda, y = -1, z = \frac{4}{3}\lambda - \frac{1}{3}$$

$$\mathbf{r} = \begin{pmatrix}3\\-1\\-\frac{1}{3}\end{pmatrix} + \lambda\begin{pmatrix}-2\\0\\\frac{4}{3}\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix} + \mu\begin{pmatrix}-3\\0\\2\end{pmatrix}$$

(using $\mu = 1.5(\lambda - 1)$ to get integer values throughout)

b When $\mu = -\frac{1}{3}, \mathbf{r} = \begin{pmatrix}2\\-1\\\frac{1}{3}\end{pmatrix}$ so $p = \frac{1}{3}$

19 $x = 0$ so $\lambda = 3$. The point is $(0, 12, 5)$ so $p = 12, q = 5$.

20 Direction vectors $\mathbf{d_1} = \begin{pmatrix}-1\\1\\2\end{pmatrix}, \mathbf{d_2} = \begin{pmatrix}2\\2\\3\end{pmatrix}$

Since $\mathbf{d_1} \neq k\mathbf{d_2}$, the two lines are not parallel.

Solving for a point of intersection:

$$\begin{cases} 1-\lambda = 7+2\mu & (1) \\ \lambda = 2+2\mu & (2) \\ 5+2\lambda = 7+3\mu & (3) \end{cases}$$

$(1)+(2)\colon 1 = 9+4\mu \Rightarrow \mu = -2$

$(2) \Rightarrow \lambda = -2$

Substituting into $(3)\colon 5-4 = 7-6$

Since this is true, the two lines intersect; the point of intersection is $(3, -2, 1)$.

21 Direction vectors $\mathbf{d_1} = \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix}, \mathbf{d_2} = \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$

Since $\mathbf{d_1} \neq k\mathbf{d_2}$, the two lines are not parallel.

Solving for a point of intersection:

$$\begin{cases} -3-\lambda = 8-\mu & (1) \\ 5-2\lambda = 5-3\mu & (2) \\ 2-4\lambda = 1+3\mu & (3) \end{cases}$$

$(2)\colon 2\lambda = 3\mu$

Substituting into $(3)\colon 2-6\mu = 1+3\mu \Rightarrow 9\mu = 1$

Then $\mu = \frac{1}{9}, \lambda = \frac{1}{6}$

Substituting into $(1)\colon -3-\lambda = -\frac{10}{6}, 5-3\mu = \frac{71}{9}$

Since these values are inconsistent with equation (1), the two lines do not intersect; since they are not parallel, they are skew.

22 a The direction vector of the first line is $\mathbf{d_1} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and, reading the denominators of the Cartesian form of the second line, $\mathbf{d_2} = \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix} = 3\mathbf{d_1}$

The two direction vectors are parallel.

Substituting $x = -1, y = 1, z = 2$, the position of the known point on the first line, into the second equation, gives

$$\frac{-1-5}{6} = \frac{1-7}{6} = \frac{2-5}{3}$$

Each of these fractions has the value -1, so this is a true statement.

The two lines are parallel and pass through the point $(-1, 1, 2)$ so must be the same line.

b Substituting $x = 4t-5, y = 4t-3, z = 1+2t$ into the Cartesian equation in part a:

$$\frac{4t}{6} = \frac{4t-10}{6} = \frac{2t-4}{3}$$

These equations are inconsistent so there can be no common point between the lines; they are parallel and distinct, not coincident.

23 a When $t = -2, \mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix} = \overrightarrow{OA}$ so A lies on the line.

When $t = 0, \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = \overrightarrow{OB}$ so B lies on the line.

b C must then be represented by $t = 2$ so that the distance between AB and BC is the same.

C has coordinates $(0, 3, 0)$

24 a $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$ so the line has vector equation $\mathbf{r} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$.

b Q is represented by $\lambda = 1$ and P by $\lambda = 0$.

Then if $PR = 3PQ$, R must be represented by $\lambda = 3$ or $\lambda = -3$.

R has coordinates $(-5, -5, 11)$ or $(19, 7, -7)$.

25 a Direction vector $\mathbf{d} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$ so the line equation is $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$.

b $\left| \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \right| = \sqrt{2^2 + (-3)^2 + 6^2} = 7$

c $AP = 35 = 5|\mathbf{d}|$ so P has position given by $\lambda = \pm 5$

P has coordinates $(12, -14, 34)$ or $(-8, 16, -26)$

26 Line AB has equation $\mathbf{r_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$.

Line CD has equation $\mathbf{r_2} = \begin{pmatrix} 8 \\ 3 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$.

Solving $\mathbf{r_1} = \mathbf{r_2}$:

$$\begin{cases} 1 + 5\lambda = 8 - 2\mu & (1) \\ \quad 5\lambda = 3 & (2) \end{cases}$$

$(2): \lambda = \frac{3}{5}$

Point of intersection is $(4, 3, 3)$

27 a The y-axis is the line where $x = z = 0$.

$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 + 2\lambda \\ 7\lambda - 1 \\ -9 - 3\lambda \end{pmatrix}$

If $x = z = 0$ then $\lambda = -3$ so the point is $(0, -22, 0)$.

b The z-axis is the line where $x = y = 0$.

As in part **a**, if $x = 0$ then $\lambda = -3$ at which point $y \neq 0$.

Hence the line does not intersect the z-axis

28 The lines have vector equations $\mathbf{r_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{r_2} = \begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix}$

Setting these equal:

$$\begin{cases} \mu = 1 + \lambda & (1) \\ 1 + 2\mu = 7 & (2) \\ \mu - 1 = \lambda p - 4 & (3) \end{cases}$$

$(2): \mu = 3$

$(1): \lambda = 2$

$(3): 2 = 2p - 4$

$p = 3$ allows the two lines to have a consistent solution (point of intersection)

The point of intersection is $(3, 7, 2)$

29 Let P, with position vector $\mathbf{r}$, be the point on the line closest to the origin.

Then $\overrightarrow{OP} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 0$

$$\left(\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 0$$

$0 + 9\lambda = 0$

$\lambda = 0$

$\overrightarrow{OP} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ so $OP = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

30 Let P be the point with position vector $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and Q the point on the line with position vector $\mathbf{r}$, which lies closest to P.

Then $\overrightarrow{PQ} \cdot \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = 0$

$$\left(\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = 0$$

$-7 + 11t = 0$

$\lambda = \frac{7}{11}$

$\overrightarrow{PQ} = \frac{1}{11} \begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$ so $PQ = \frac{1}{11}\sqrt{1^2 + (-4)^2 + 7^2} = \frac{\sqrt{66}}{11} \approx 0.739$

Exercise 2C

13 a $|\mathbf{v}| = \sqrt{0.5^2 + 2^2 + 1.5^2} = \sqrt{6.5} = 2.55\text{ m s}^{-1}$

b $\mathbf{r} = (12\mathbf{i} - 5\mathbf{j} + 11\mathbf{k}) + t(0.5\mathbf{i} + 2\mathbf{j} + 1.5\mathbf{k})$

c Solving for $\mathbf{r} = (16\mathbf{i} + 8\mathbf{j} + 14\mathbf{k})$:

$$\begin{cases} 12 + 0.5t = 16 & (1) \\ -5 + 2t = 8 & (2) \\ 11 + 1.5t = 14 & (3) \end{cases}$$

$4(1) - (2)\text{: } 53 = 56$

This shows that the system is consistent, so the particle does not pass through the point $(16, 8, 14)$

14 At $t = 3, \mathbf{r_1} = \begin{pmatrix} 13 \\ 2 \\ 9 \end{pmatrix}$ and $\mathbf{r_2} = \begin{pmatrix} 0.5 \\ 6 \\ -1.5 \end{pmatrix}$

Then $|\mathbf{r_1} - \mathbf{r_2}| = \sqrt{12.5^2 + (-4)^2 + 10.5^2} = \sqrt{282.5} \approx 16.8\text{ m}$

15 a At $t = 0$, the position of the object is $(3, -1, 4)$

b Speed is $\left|\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}\right| = \sqrt{11} \approx 3.32\text{ m s}^{-1}$

c $\mathbf{r}(3) = \begin{pmatrix} 6 \\ -4 \\ 13 \end{pmatrix}$

so the distance from the origin is $\sqrt{6^2 + (-4)^2 + 13^2} = \sqrt{221} \approx 14.9\text{ m}$

16 a

$\mathbf{r}_A = \begin{pmatrix} 9 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\mathbf{r}_B = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 6 \\ -2 \end{pmatrix}$

b

$$\begin{aligned} |\boldsymbol{r_A}(2) - \boldsymbol{r_B}(2)| &= \left|\begin{pmatrix} 17 - 9 \\ 11 - (-2) \end{pmatrix}\right| \\ &= \sqrt{8^2 + 13^2} \\ &= 15.3\text{ km} \end{aligned}$$

c Require $9 + 4t = -3 + 6t$

$2t = 12$

$t = 6$

A is at $(33, 31)$ and B is at $(33, -10)$ after 6 hours, so A is due north of B.

17 Direction vector is $\mathbf{d} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$|\mathbf{d}| = \sqrt{9} = 3$ so the vector position of the aeroplane at time t is $\mathbf{r}$

$$= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t\left(\frac{894}{3}\right)\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 596 \\ -596 \\ 298 \end{pmatrix}$$

18 a Direction vectors $\mathbf{d_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \mathbf{d_2} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Since $\mathbf{d_1} \neq k\mathbf{d_2}$, the two lines are not parallel.

Solving for a point of intersection:

$$\begin{cases} 1 + \lambda = 1 + 2\mu & (1) \\ -\lambda = -1 - \mu & (2) \\ 3 + 2\lambda = 4 + 3\mu & (3) \end{cases}$$

$(1) + (2)$: $1 = \mu$ so $\lambda = 2$

Substituting into (3): $7 = 7$

This is true, so the two lines have a point of intersection at $(3, -2, 7)$.

b Although they both pass through the same point, the first line does so at $t = 2$ and the second at $t = 1$ so (taking the two helicopters as having point locations – that is, assuming they are small enough given the scale of the model) they will not collide.

19 a Re-expressing the two particle positions with different time parameters λ and μ:

$$\mathbf{r_1} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}, \mathbf{r_2} = \begin{pmatrix} 9 \\ -2 \\ 22 \end{pmatrix} + \mu\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$v_1 = \left|\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}\right| = \sqrt{54}$$

$$v_2 = \left|\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}\right| = \sqrt{9} = 3$$

b Solving for a point of intersection:

$$\begin{cases} 1 + 2\lambda = 9 - 2\mu & (1) \\ -3 + \lambda = -2 + \mu & (2) \\ 3 + 7\lambda = 22 + 2\mu & (3) \end{cases}$$

$(1) + (3)$: $4 + 9\lambda = 31 \Rightarrow \lambda = 3, \mu = 1$

Substituting into (2): $-3 + 3 = -2 + 2$ is true, so the system is consistent and the two paths intersect, at point $(7, 0, 24)$.

However, the two particles are not at the intersection point at the same time; the first particle is at the intersection point at $t = 3$ and the second particle is there at $t = 1$, so they do not meet.

20 a $\mathbf{r}_P = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \mathbf{r}_Q = \begin{pmatrix} 7 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

b Solving for a point of intersection, using λ for the parameter in $\mathbf{r}_P$ and μ for the parameter in $\mathbf{r}_Q$:

$$\begin{cases} -1 + \lambda = 7 + 2\mu & (1) \\ -1 + 4\lambda = -2 - 3\mu & (2) \\ 2 + 3\lambda = 5 - \mu & (3) \end{cases}$$

$4(1) - (2): -3 = 30 + 11\mu \Rightarrow \mu = -3$

$(1): \lambda = 8 + 2\mu = 2$

Substituting into (3): $2 + 6 = 5 - (-3)$ is true, so the system is consistent and the two paths intersect, at point $(1, 7, 8)$.

However, the two particles are not at the intersection point at the same time; the first particle is at the intersection point at $t = 2$ and the second particle is there at $t = -3$, so they do not meet.

21 a $\mathbf{r}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 5 \end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

b $\mathbf{r}_1 - \mathbf{r}_2 = \begin{pmatrix} 3 - 6t \\ -5 + 4t \end{pmatrix}$

Distance $d = |\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{(3 - 6t)^2 + (-5 + 4t)^2} = \sqrt{52t^2 - 76t + 34}$

c From the GDC, the minimum value for d is $d\left(\frac{19}{26}\right) = 2.50$ miles

d Require that $0 + 5t = 5 + t \Rightarrow t = 1.25$ hours

22 a $\mathbf{r}_\mathrm{A} = \begin{pmatrix} 12 \\ -10 \\ 1 \end{pmatrix} + t\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$

$\mathbf{r}_\mathrm{B} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

b $\boldsymbol{r_A} - \boldsymbol{r_B} = \begin{pmatrix} 7 \\ -11 \\ -3 \end{pmatrix} + t\begin{pmatrix} -6 \\ 3 \\ 0 \end{pmatrix}$

$$\begin{aligned} d &= |\boldsymbol{r_A} - \boldsymbol{r_B}| \\ &= \sqrt{(7 - 6t)^2 + (-11 + 3t)^2 + 9} \\ &= \sqrt{45t^2 - 150t + 179} \end{aligned}$$

c

$$\begin{aligned} d^2 &= 45t^2 - 150t + 179 \\ &= 45\left(t^2 - \frac{10}{3}t\right) + 179 \\ &= 45\left[\left(t - \frac{5}{3}\right)^2 - \frac{25}{9}\right] + 179 \\ &= 45\left(t - \tfrac{5}{3}\right)^2 + 54 \end{aligned}$$

So minimum d is $\sqrt{54} \approx 7.35\,\mathrm{m}$

23 a i If they collide then $\mathbf{r_1} = \mathbf{r_2}$

$$\begin{pmatrix}2\\4\\3\end{pmatrix} + t\begin{pmatrix}4\\5\\-2\end{pmatrix} = \begin{pmatrix}5\\1\\0\end{pmatrix} + t\begin{pmatrix}-1\\10\\3\end{pmatrix}$$

$$\begin{cases} 2 + 4t = 5 - t & (1)\\ 4 + 5t = 1 + 10t & (2)\\ 3 + 2t = 3t & (3)\end{cases}$$

All of these have the solution $t = \frac{3}{5}$, so there is a single consistent solution indicating that the two drones will collide.

ii $\mathbf{r_1}\left(\frac{3}{5}\right) = \mathbf{r_2}\left(\frac{3}{5}\right) = \begin{pmatrix}4.4\\7\\1.8\end{pmatrix}$

b $\mathbf{r_2}\left(\frac{1}{2}\right) = \begin{pmatrix}4.5\\6\\1.5\end{pmatrix}$

Then $\mathbf{r_2}(t) = \begin{pmatrix}4.5\\6\\1.5\end{pmatrix} + \left(t - \frac{1}{2}\right)\begin{pmatrix}0\\8\\2\end{pmatrix}$ for $t \geq \frac{1}{2}$

$$\mathbf{r_2}\left(\frac{3}{5}\right) = \begin{pmatrix}4.5\\6\\1.5\end{pmatrix} + \frac{1}{10}\begin{pmatrix}0\\8\\2\end{pmatrix} = \begin{pmatrix}4.5\\6.8\\1.7\end{pmatrix}$$

Distance between expected crash point and this position is therefore

$$\begin{aligned}\left|\mathbf{r_1}\left(\frac{2}{3}\right) - \mathbf{r_2}\left(\frac{2}{3}\right)\right| &= \sqrt{0.1^2 + 0.2^2 + 0.1^2}\\ &= 0.1\sqrt{6}\\ &\approx 0.245\end{aligned}$$

24 a One is vertically above the other when **i** and **j** components match

$$\begin{pmatrix}1.2t\\0.36 + 0.8t\end{pmatrix} = \begin{pmatrix}3.96 - t\\t\end{pmatrix}$$

Solution $t = 1.8$ is consistent for both components.

b The height difference will be the difference in **k** components at that time.

$$\mathbf{r_1}(1.8) = \begin{pmatrix}2.16\\1.8\\3.12\end{pmatrix}, \mathbf{r_2}(1.8) = \begin{pmatrix}2.16\\1.8\\1.54\end{pmatrix}$$

So $|\mathbf{r_1} - \mathbf{r_2}| = 1.58\,\text{m}$

25 a $\mathbf{r_B} = t\begin{pmatrix}64\\0\\0\end{pmatrix}, \mathbf{r_S} = \begin{pmatrix}0\\0.5\\-0.02\end{pmatrix} + kt\begin{pmatrix}40\\-25\\c\end{pmatrix}$

If the two vessels coincide at a time t then

$$\begin{cases} 64t = 40kt & (1)\\ 0 = 0.5 - 25kt & (2)\\ 0 = -0.02 + ckt & (3)\end{cases}$$

(1): $k = 1.6$

(2): $0.5 = 40t \Rightarrow t = 0.0125$

$$(3): c = \frac{0.02}{kt} = 1$$

b The velocity of the submarine is $k\begin{pmatrix} 40 \\ -25 \\ c \end{pmatrix} = \begin{pmatrix} 64 \\ -40 \\ 1.6 \end{pmatrix}$ so the speed is $75.5\ \mathrm{km\ h^{-1}}$

Exercise 2D

33 $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix}$

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{OA}}{|AB||OA|} = \frac{(-6 + 10 - 3)}{\sqrt{35}\sqrt{17}} = \frac{1}{\sqrt{595}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{595}}\right) = 87.7°$$

34 $\overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}, \overrightarrow{BD} = \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix}$

$$\cos\theta = \frac{\overrightarrow{AC} \cdot \overrightarrow{BD}}{|AC||BD|} = \frac{24 + 0 - 1}{\sqrt{17}\sqrt{53}} = \frac{23}{\sqrt{901}}$$

$$\theta = \cos^{-1}\left(\frac{23}{\sqrt{901}}\right) = 40.0°$$

35 $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{10}{15} = \frac{2}{3}$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 48.2°$$

36 $\cos\theta = \frac{\mathbf{c}\cdot\mathbf{d}}{|\mathbf{c}||\mathbf{d}|} = \frac{-15}{9\times 12} = -\frac{5}{36}$

$$\theta = \cos^{-1}\left(-\frac{5}{36}\right) = 98.0°$$

37 $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$$\frac{1}{2} = \frac{12}{8|\mathbf{b}|}$$

$$|\mathbf{b}| = 3$$

38 The lines have direction vectors $\mathbf{d_1} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

Acute angle θ between the lines is such that

$$\cos\theta = \left|\frac{\mathbf{d_1} \cdot \mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|}\right| = \left|\frac{-2}{\sqrt{11}\sqrt{18}}\right|$$

$$\theta = 81.8°$$

39 The lines have direction vectors $\mathbf{d_1} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\mathbf{d_1} \cdot \mathbf{d_2} = 4 - 1 - 3 = 0$ so the two lines have perpendicular vectors.

Both lines pass through the same point $(0, 0, 1)$ so they are perpendicular within a single plane.

Comment: Remember to show or observe that the two vectors are coplanar. While it might be argued that any two lines with perpendicular direction vectors should be considered perpendicular, general usage would say that if the lines do not intersect, they are simply skew and **not** truly perpendicular.

Any pair of parallel lines are necessarily coplanar, and it is standard to also require that two lines must cross at 90° angle (that is, must intersect and so be coplanar) to be considered perpendicular.

40 $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta = 17.5$

41 $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

$7 = 2 \times 5 \sin\theta$

$\theta = \sin^{-1}(0.7) \approx 0.775$

42 $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

$$\left|\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}\right| = 7 \times 1 \sin\theta$$

$\sqrt{17} = 7 \sin\theta$

$$\theta = \sin^{-1}\left(\frac{\sqrt{17}}{7}\right) \approx 0.630$$

43 $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -8 \\ -5 \\ 1 \end{pmatrix}$

A vector perpendicular to both **a** and **b** is $-8\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

44 **a** $\left|\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right| = \sqrt{13}$ so the river velocity is $\frac{10}{\sqrt{13}}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b

$$\frac{\frac{10}{\sqrt{13}}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right|} = \frac{10}{\sqrt{13}} \times \frac{11}{\sqrt{17}} = \frac{110}{\sqrt{221}} = 7.40\text{ km h}^{-1}$$

45 Position vectors of the vertices are $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

So $\mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \mathbf{c} - \mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}, \mathbf{b} - \mathbf{c} = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$

$$\cos A = \frac{(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}||\mathbf{c} - \mathbf{a}|} = \frac{4 + 0 + 2}{\sqrt{9}\sqrt{17}} = \frac{2}{\sqrt{17}} \Rightarrow A = 61.0°$$

$$\cos B = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})}{|\mathbf{b}-\mathbf{a}||\mathbf{b}-\mathbf{c}|} = \frac{-3+4+2}{\sqrt{9}\sqrt{14}} = \frac{1}{\sqrt{14}} \Rightarrow B = 74.5°$$

$C = 180° - A - B = 44.5°$

46 $\mathbf{b}-\mathbf{a} = \begin{pmatrix}2\\-2\\3\end{pmatrix}, \mathbf{c}-\mathbf{a} = \begin{pmatrix}5\\0\\-4\end{pmatrix}, \mathbf{b}-\mathbf{c} = \begin{pmatrix}-3\\-2\\7\end{pmatrix}$

$$\cos A = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}||\mathbf{c}-\mathbf{a}|} = \frac{10+0-12}{\sqrt{17}\sqrt{41}} = -\frac{2}{\sqrt{697}} \Rightarrow A = 94.3°$$

$$\cos B = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})}{|\mathbf{b}-\mathbf{a}||\mathbf{b}-\mathbf{c}|} = \frac{-6+4+21}{\sqrt{17}\sqrt{62}} = \frac{19}{\sqrt{1054}} \Rightarrow B = 54.2°$$

$C = 180° - A - B = 31.5°$

47 **a** $\mathbf{a} = \begin{pmatrix}1\\-2\\2\end{pmatrix}, \mathbf{b} = \begin{pmatrix}3\\-1\\7\end{pmatrix}, \mathbf{c} = \begin{pmatrix}5\\0\\0\end{pmatrix}$

$\mathbf{b}-\mathbf{a} = \begin{pmatrix}2\\1\\5\end{pmatrix}, \mathbf{c}-\mathbf{a} = \begin{pmatrix}4\\2\\-2\end{pmatrix}, \mathbf{b}-\mathbf{c} = \begin{pmatrix}-2\\-1\\7\end{pmatrix}$

$$\cos A = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}||\mathbf{c}-\mathbf{a}|} = \frac{8+2-10}{\sqrt{30}\sqrt{24}} = 0 \Rightarrow A = 90°$$

b $\cos B = \frac{(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c})}{|\mathbf{b}-\mathbf{a}||\mathbf{b}-\mathbf{c}|} = \frac{-4-1+35}{\sqrt{30}\sqrt{54}} = \frac{\sqrt{5}}{3} \Rightarrow B = 41.8°$

$C = 180° - A - B = 48.2°$

c $\text{Area} = \frac{1}{2}(AB)(AC) = \frac{1}{2}\sqrt{30}\sqrt{24} = 6\sqrt{5}$

48

$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$$

$$\frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{1\times|\mathbf{q}|}$$

$|\mathbf{q}| = 6$

49 Require $\begin{pmatrix}4+2t\\-1+t\\2+t\end{pmatrix}\cdot\begin{pmatrix}3\\5\\1\end{pmatrix} = 0$

$12+6t-5+5t+2+t = 0$

$12t = -9$

$t = -\frac{3}{4}$

50 Require $\begin{pmatrix}t\\0\\-3\end{pmatrix}\cdot\begin{pmatrix}2t\\1\\t\end{pmatrix} = 0$

$2t^2 - 3t = 0$

$t = 0, \frac{3}{2}$

51 a $\mathbf{r} = \begin{pmatrix} 0.5 \\ 7 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1.5 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -8 \end{pmatrix}$

(using $\mu = 2(\lambda - 1)$ to get integer values throughout)

b Direction vector of the line is $\mathbf{d} = \begin{pmatrix} 3 \\ 0 \\ -8 \end{pmatrix}$

Angle with $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is θ where

$\cos\theta = \frac{\mathbf{d} \cdot \mathbf{i}}{|\mathbf{d}|} = \frac{3}{\sqrt{73}}$

$\theta = 69.4°$

52 $\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = 17$

$\left| \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} \right| = \sqrt{30}$

$\left| \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right| = \sqrt{10}$

$\theta = \cos^{-1}\left(\frac{17}{\sqrt{30}\sqrt{10}}\right) = 11.0°$

53 $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\left| \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right| = \sqrt{2}$ so a unit vector perpendicular to the two vectors would be $\frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

54 $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

$\left| \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \right| = \sqrt{14}$

Unit vector $= \frac{\sqrt{14}}{14} \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

54 $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

$\left| \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \right| = \sqrt{14}$

A unit vector perpendicular to both $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ is $\frac{\sqrt{14}}{14} \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$.

55 **a** $\mathbf{a} \times \mathbf{b}$ gives a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

The scalar product of two perpendicular vectors is always zero.

Taking the scalar product of $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a}$ must therefore produce the value zero.

b Distributing:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} \\ &= 0 - 0 \qquad \text{(using reasoning from part } \mathbf{a}\text{)} \\ &= 0\end{aligned}$$

56 **a** $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -4 \\ 6 \\ 2 \end{pmatrix}, \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \begin{pmatrix} -8 \\ -6 \\ 1 \end{pmatrix}$

$$\mathbf{p} = \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix} \times \begin{pmatrix} -4 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ -12 \\ 72 \end{pmatrix}$$

$$\mathbf{q} = \begin{pmatrix} -4 \\ -12 \\ -1 \end{pmatrix} \times \begin{pmatrix} -8 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} -18 \\ 12 \\ -72 \end{pmatrix}$$

b $\mathbf{p} = -\mathbf{q}$

57 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix} = \overrightarrow{DC}$

$$\overrightarrow{BC} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} = \overrightarrow{AD}$$

The quadrilateral has two pairs of parallel (and equal length) sides, so is a parallelogram

b A parallelogram with side vectors $\mathbf{u}$ and $\mathbf{v}$ has area $|\mathbf{u} \times \mathbf{v}|$

$$\text{Area } ABCD = \left| \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} \right| = \left| \begin{pmatrix} -41 \\ -10 \\ -6 \end{pmatrix} \right| = \sqrt{1817} \approx 42.6$$

58 $A(2, 1, 2), B(5, 0, 1)$ and $C(-1, 3, 5)$.

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$$

A triangle with side vectors $\mathbf{u}$ and $\mathbf{v}$ has area $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$

$$\text{Area} = \frac{1}{2}\left| \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} \right| = \frac{1}{2}\left| \begin{pmatrix} -1 \\ -6 \\ 3 \end{pmatrix} \right| = \frac{1}{2}\sqrt{46} \approx 3.39$$

59 **a** $\overrightarrow{BC} = \begin{pmatrix} 8 \\ -3 \\ -2 \end{pmatrix} = \overrightarrow{AD}$ so D has coordinates $(11, -2, 0)$

b A parallelogram with side vectors $\mathbf{u}$ and $\mathbf{v}$ has area $|\mathbf{u} \times \mathbf{v}|$

$$\overrightarrow{AB} = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} = \overrightarrow{DC}$$

$$\text{Area } ABCD = \left|\begin{pmatrix}8\\-3\\-2\end{pmatrix}\times\begin{pmatrix}-4\\0\\3\end{pmatrix}\right| = \left|\begin{pmatrix}-9\\-16\\-12\end{pmatrix}\right| = \sqrt{481} \approx 21.9$$

60 $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\cos\theta$

$|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

where θ is the angle between vectors **a** and **b**

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}|^2 + (\mathbf{a}\cdot\mathbf{b})^2 &= (|\mathbf{a}||\mathbf{b}|)^2(\cos^2\theta + \sin^2\theta)\\ &= |\mathbf{a}|^2|\mathbf{b}|^2\end{aligned}$$

61 The vector of windspeed is $\frac{3}{\left|\begin{pmatrix}3\\4\end{pmatrix}\right|}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}1.8\\2.4\end{pmatrix}$

Magnitude in the direction of the track:

$$\frac{\begin{pmatrix}1.8\\2.4\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}}{\left|\begin{pmatrix}1\\2\end{pmatrix}\right|} = \frac{6.6}{\sqrt{5}} = 2.68 > 2$$

The record time will not be valid.

62 a $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = \overrightarrow{AD} - \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

b

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{b}-\mathbf{a}) &= \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{a}\\ &= \mathbf{b}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{a}\\ &= |\mathbf{b}|^2 - |\mathbf{a}|^2\end{aligned}$$

c If $ABCD$ is a rhombus then $|\mathbf{a}| = |\mathbf{b}|$ so $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{b}-\mathbf{a}) = 0$

That is, $AC \perp BD$, the diagonals are perpendicular.

63 a If A and B have position vectors **a** and **b** respectively, then $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b} = \lambda\mathbf{a}$

By definition, if $\overrightarrow{OB} = \lambda\overrightarrow{OA}$ then O, A and B are collinear.

b $\overrightarrow{BA} = (1-\lambda)\begin{pmatrix}2\\1\\4\end{pmatrix}$

$\overrightarrow{CB} = \begin{pmatrix}2\lambda - 12\\ \lambda - 2\\ 4\lambda - 4\end{pmatrix}$

Require $\overrightarrow{BA}\cdot\overrightarrow{CB} = 0$

$(1-\lambda)(4\lambda - 24 + \lambda - 2 + 16\lambda - 16) = 0$

$(1-\lambda)(21\lambda - 42) = 0$

$\lambda = 2$

($\lambda = 1$ represents the degenerate case where A and B are collocated, which does not represent a solution to the problem posed).

c B is the point on extended line OA for which the distance from C to the line is shortest.

$\overrightarrow{CB} = \begin{pmatrix} -8 \\ 0 \\ 4 \end{pmatrix}$ so $CB = \sqrt{80} = 4\sqrt{5}$

64 a Require $\left(\begin{pmatrix} 21 \\ 5 \\ 10 \end{pmatrix} - \mathbf{r}\right) \cdot \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} = 0$

$\begin{pmatrix} 16 - 2\lambda \\ 4 + 3\lambda \\ 8 - 3\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} = 0$

$44 - 22\lambda = 0$

$\lambda = 2$

M has coordinates $(9, -5, 8)$

b When $\lambda = 5$, $\mathbf{r} = \begin{pmatrix} 15 \\ -15 \\ 17 \end{pmatrix}$, so Q lies on the line l.

c $\overrightarrow{MQ} = 3\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$

If $|PR| = |PQ|$ then $\overrightarrow{MR} = -3\begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$ by symmetry about the perpendicular PM.

R has coordinates $(3, 4, -1)$

65 a $C(5,4,0), F(5,0,2), G(5,4,2), H(0,4,2)$

b $\overrightarrow{BE} = \begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix}$ and $\overrightarrow{BG} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$

$$\begin{aligned} \text{Area } BEG &= \frac{1}{2}\left|\overrightarrow{BE} \times \overrightarrow{BG}\right| \\ &= \frac{1}{2}\left|\begin{pmatrix} -8 \\ 10 \\ -20 \end{pmatrix}\right| \\ &= \frac{1}{2}\sqrt{564} \approx 11.9 \end{aligned}$$

Mixed Practice 2

1 a Total displacement from start: $\begin{pmatrix} -22 \\ -5 \\ 8 \end{pmatrix} + \begin{pmatrix} 215 \\ -73 \\ -0.5 \end{pmatrix} + \begin{pmatrix} 83 \\ 16 \\ 0 \end{pmatrix} = \begin{pmatrix} 276 \\ -62 \\ 7.5 \end{pmatrix}$

To return to the start, the plane must travel $\begin{pmatrix} -276 \\ 62 \\ -7.5 \end{pmatrix}$

b $\left|\begin{pmatrix} -276 \\ 62 \\ -7.5 \end{pmatrix}\right| = \sqrt{(-276)^2 + 62^2 + (-7.5)^2} = \sqrt{80076.25} \approx 283$ km

2 **a** $\overrightarrow{MD} = \frac{1}{2}\mathbf{b} - \mathbf{a}$

b

$$\begin{aligned}\overrightarrow{AN} &= \overrightarrow{AM} + \overrightarrow{MN}\\ &= \mathbf{a} + \frac{1}{2}\mathbf{b} + \frac{1}{2}\overrightarrow{MD}\\ &= \frac{1}{2}\mathbf{a} + \frac{3}{4}\mathbf{b}\end{aligned}$$

c $\overrightarrow{AP} = \mathbf{a} + \frac{3}{2}\mathbf{b} = 2\overrightarrow{AN}$

This shows that A, P and N are collinear, with P the midpoint of AN.

3 **a** $\overrightarrow{AD} = \begin{pmatrix} 2 \\ 0 \\ k-7 \end{pmatrix}$

b $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix}$

If the two vectors are perpendicular then $\begin{pmatrix} 2 \\ 0 \\ k-7 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} = 0$

$-4 - 4k + 28 = 0$

$k = 6$

c $\overrightarrow{BC} = 2\overrightarrow{AD} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$

C has coordinates $(3, 6, 1)$

d

$$\begin{aligned}\cos(A\hat{D}C) &= \frac{(\overrightarrow{DA} \cdot \overrightarrow{DC})}{|AD||DC|}\\ &= \frac{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 5 \\ -5 \end{pmatrix}}{\sqrt{5}\sqrt{50}}\\ &= \frac{-5}{5\sqrt{10}}\\ &= -\frac{1}{\sqrt{10}}\end{aligned}$$

4 Point of intersection occurs where $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 9 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

$$\begin{cases} 5 - t = 5 + 2s & (1) \\ 1 + t = 4 + s & (2) \\ 2 + 3t = 9 + s & (3) \end{cases}$$

$(1) + (2){:}\ 6 = 9 + 3s \Rightarrow s = -1$

$(2){:}\ t = 3 + s = 2$

Substituting into (3):

$2 + 3t = 8 = 9 + s$

The system is consistent for $s = -1, t = 2$ so there is an intersection point, at $(3, 3, 8)$

5 **a** The direction vectors are not multiples of each other, so the lines are not parallel.

$\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 4 + 2 - 6 = 0$ so the two lines directions are perpendicular.

b Solving for an intersection:

$$\begin{cases} 2 + 4\lambda = -2 + \mu & (1) \\ -1 - \lambda = -2\mu & (2) \\ 5 + 2\lambda = 3 - 3\mu & (3) \end{cases}$$

$2(1) + (2): 3 + 7\lambda = -4 \Rightarrow \lambda = -1$

$(2): \mu = \frac{1}{2}(1 + \lambda) = 0$

Substituting into $(3): 5 + 2\lambda = 3 = 3 - 3\mu$

The system is consistent so the two lines do intersect and are perpendicular.

6 Solve for an intersection:

$$\begin{cases} 4 - 3\lambda = 1 + 4\mu & (1) \\ 1 + 3\lambda = -2 + 3\mu & (2) \\ 2 + \lambda = 0.5 + 2\mu & (3) \end{cases}$$

$(1) + (2): 5 = -1 + 7\mu \Rightarrow \mu = \frac{6}{7}$

$(1): \lambda = \frac{3 - 4\mu}{3} = -\frac{1}{7}$

Substituting into (3):

$2 + \lambda = \frac{13}{7} \neq 0.5 + 2\mu = \frac{31}{14}$

Since the system is inconsistent, the two lines do not intersect.

7 **a** Direction $\mathbf{d_1} = \overrightarrow{AB} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

A vector equation of the line is $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$

b $\mathbf{d_2} = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$

Angle θ between the lines is such that

$\cos\theta = \frac{\mathbf{d_1} \cdot \mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|} = -\frac{1}{\sqrt{27}\sqrt{51}}$

$\theta = 88.5°$

c Require $\begin{pmatrix}4\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\7\end{pmatrix}=\begin{pmatrix}7\\3\\k\end{pmatrix}$

$\lambda = 3$ so $k = 24$

d $AC = \sqrt{6^2+6^2+22^2} = \sqrt{556} \approx 23.6$

8 **a** $\mathbf{r} = \begin{pmatrix}1\\4\\-1\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\3\end{pmatrix}$

b Direction $\mathbf{d} = \begin{pmatrix}3\\-2\\3\end{pmatrix}$

$|\mathbf{d}| = \sqrt{3^2+(-2)^2+3^2} = \sqrt{22}$

The unit vector in this direction is $\frac{\sqrt{22}}{22}\begin{pmatrix}3\\-2\\3\end{pmatrix}$

9 **a** $|\mathbf{v}| = \sqrt{116^2+52^2+12^2} = \sqrt{16304} \approx 128\ \mathrm{m\,s^{-1}}$

b The z coordinate will equal 1000 at time $t = \frac{1000}{12} \approx 83.3$ s

10 a $\mathbf{r}_{\boldsymbol{A}}(t) = \begin{pmatrix}2\\-8\end{pmatrix}+t\begin{pmatrix}-5\\3\end{pmatrix}$

$\mathbf{r}_{\boldsymbol{B}}(t) = \begin{pmatrix}3\\4\end{pmatrix}+t\begin{pmatrix}2\\-6\end{pmatrix}$

b Distance between the ships is $d(t) = |\mathbf{r}_{\boldsymbol{A}} - \mathbf{r}_{\boldsymbol{B}}|$

$d(t) = \left|\begin{pmatrix}-1\\-12\end{pmatrix}+t\begin{pmatrix}-7\\9\end{pmatrix}\right|$

$d(3) = \left|\begin{pmatrix}-22\\15\end{pmatrix}\right| = \sqrt{(-22)^2+15^2} = \sqrt{709} \approx 26.6$ km

c Require that the **j** components match:

$-8+3t = 4-6t \Rightarrow 9t = 12$

$t = \frac{4}{3}$ hour $= 80$ minutes

11 **a** $\mathbf{r}(t) = \begin{pmatrix}2\\-3\\0\end{pmatrix}+t\begin{pmatrix}10\\-9\\7\end{pmatrix}$

b Speed $= \left|\begin{pmatrix}10\\-9\\7\end{pmatrix}\right| = \sqrt{230} \approx 15.2\ \mathrm{m\,s^{-1}}$

c Require $\mathbf{r} = \begin{pmatrix}67\\-61.5\\z\end{pmatrix}$

$2+10t = 67 \Rightarrow t = 6.5$

$\mathbf{r}(6.5) = \begin{pmatrix}67\\-61.5\\45.5\end{pmatrix}$ so at 6.5 s after launch, the drone is 45.5 m directly above the controller.

12 $\begin{pmatrix} 3\sin x \\ 8 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4\cos x \\ 1 \\ -2 \end{pmatrix} = 12\sin x\cos x + 6$

$2\sin x\cos x = -1$

$\sin 2x = -1$

$2x = \frac{3\pi}{2} + 2n\pi$

$x = \frac{3\pi}{4} + n\pi$

The solution in the given interval is $\frac{3\pi}{4}$

13 a $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 4 \\ -1 \\ p \end{pmatrix}$

$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -3 \\ 7 \\ 16 \end{pmatrix}$

$\begin{pmatrix} -3 \\ 7 \\ 16 \end{pmatrix} \cdot \mathbf{c} = -18$

b Require $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = 0$

$\begin{pmatrix} -3 \\ 7 \\ 16 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ p \end{pmatrix} = -19 + 16p$

$p = \frac{19}{16}$

14 a $\mathbf{a} \times \mathbf{c} = \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -2 \end{pmatrix}$

b $\mathbf{b} = \mathbf{a} + \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ so B has coordinates $(1, 2, -2)$

c Area $OABC = |\mathbf{a} \times \mathbf{c}| = \left|\begin{pmatrix} 6 \\ -5 \\ -2 \end{pmatrix}\right| = \sqrt{65}$

15 a $\left|\begin{pmatrix} -2 \\ 21 \\ 15 \end{pmatrix}\right| = \sqrt{(-2)^2 + 21^2 + 15^2} = \sqrt{670} \approx 25.9\text{ m s}^{-1}$

b The component in the aimed direction is

$$\frac{\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 21 \\ 15 \end{pmatrix}}{\left|\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}\right|} = \frac{129}{5} = 25.8\text{ m s}^{-1}$$

c The component perpendicular to the aimed direction is
$\sqrt{670 - 25.8^2} = \sqrt{4.36} \approx 2.09\text{ m s}^{-1}$

16 a i $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$

ii

$$\begin{aligned}\overrightarrow{BM} &= \overrightarrow{BA} + \overrightarrow{AM}\\ &= (\mathbf{a} - \mathbf{b}) + \frac{1}{2}(\mathbf{c} - \mathbf{a})\\ &= \frac{1}{2}\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}\end{aligned}$$

b i

$$\begin{aligned}\overrightarrow{RA} &= -\overrightarrow{AR}\\ &= -\frac{1}{3}(\mathbf{b} - \mathbf{a})\\ &= \frac{1}{3}(\mathbf{a} - \mathbf{b})\end{aligned}$$

ii

$$\begin{aligned}\overrightarrow{RT} &= \frac{2}{3}\overrightarrow{RS}\\ &= \frac{2}{3}\left(\overrightarrow{RA} + \overrightarrow{AS}\right)\\ &= \frac{2}{3}\left(\frac{1}{3}(\mathbf{a} - \mathbf{b}) + \frac{2}{3}(\mathbf{c} - \mathbf{a})\right)\\ &= -\frac{1}{9}\mathbf{a} - \frac{2}{9}\mathbf{b} + \frac{4}{9}\mathbf{c}\end{aligned}$$

c

$$\begin{aligned}\overrightarrow{BT} &= \overrightarrow{BR} + \overrightarrow{RT}\\ &= \frac{2}{3}\overrightarrow{BA} + \overrightarrow{RT}\\ &= \frac{2}{3}(\mathbf{a} - \mathbf{b}) - \frac{2}{9}\mathbf{a} - \frac{2}{9}\mathbf{b} + \frac{4}{9}\mathbf{c}\\ &= \frac{4}{9}\mathbf{a} - \frac{8}{9}\mathbf{b} + \frac{4}{9}\mathbf{c}\\ &= \frac{8}{9}\overrightarrow{BA}\end{aligned}$$

17

$$\begin{aligned}\text{Resultant } \mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} &= \begin{pmatrix}5\\0\\a\end{pmatrix} + \begin{pmatrix}b\\-7\\-10\end{pmatrix} + \begin{pmatrix}8\\c\\-2\end{pmatrix}\\ &= \begin{pmatrix}13+b\\c-7\\a-12\end{pmatrix}\end{aligned}$$

$$\mathbf{F_1} + \mathbf{F_2} = \begin{pmatrix}5\\0\\a\end{pmatrix} + \begin{pmatrix}b\\-7\\-10\end{pmatrix} = \begin{pmatrix}5+b\\-7\\a-10\end{pmatrix}$$

$$\begin{pmatrix}13+b\\c-7\\a-12\end{pmatrix} = 3\begin{pmatrix}5+b\\-7\\a-10\end{pmatrix} = \begin{pmatrix}15+3b\\-21\\3a-30\end{pmatrix}$$

$13 + b = 15 + 3b \Rightarrow b = -1$

$c - 7 = -21 \Rightarrow c = -14$

$a - 12 = 3a - 30 \Rightarrow a = 9$

18 a $\left|\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right| = \sqrt{6}$ so the velocity $\mathbf{v} = \frac{30}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10\sqrt{6} \\ -5\sqrt{6} \\ 5\sqrt{6} \end{pmatrix}$ mph

b After $\frac{5}{60}$ hours, its position relative to P is $\begin{pmatrix} \frac{5\sqrt{6}}{6} \\ -\frac{5\sqrt{6}}{12} \\ \frac{5\sqrt{6}}{12} \end{pmatrix}$

19 a $\overrightarrow{AB} = \begin{pmatrix} k-2 \\ 0 \\ 2k-1 \end{pmatrix} = \overrightarrow{DC}$

$\overrightarrow{AD} = \begin{pmatrix} 4 \\ 2k \\ 2 \end{pmatrix} = \overrightarrow{CB}$

None of these vectors can be the zero vector so, by definition, $ABCD$ is a parallelogram.

b $k = 1 \Rightarrow \overrightarrow{AB} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$

Angle A is such that $\cos B\hat{A}D = \frac{\overrightarrow{AB} \cdot \overrightarrow{AD}}{|\overrightarrow{AB}||\overrightarrow{AD}|} = \frac{-2}{\sqrt{2}\sqrt{24}}$

$B\hat{A}D = 107° = B\hat{C}D$

$A\hat{B}C = A\hat{D}C = 73.2°$

c If $ABCD$ is a rectangle then $\overrightarrow{AB} \cdot \overrightarrow{AD} = 0$

$4(k-2) + 2(2k-1) = 0$

$8k - 10 = 0$

$k = \frac{5}{4}$

20 a $x = \frac{1}{2} + 2\lambda, y = 3\lambda - 2, y = \frac{4}{3} - 2\lambda$

b The x-axis is the line where $y = z = 0$.

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + 2\lambda \\ 3\lambda - 2 \\ \frac{4}{3} - 2\lambda \end{pmatrix}$$

If $y = z = 0$ then $3\lambda - 2 = 0$ and $\frac{4}{3} - 2\lambda = 0$. These have a consistent solution $\lambda = \frac{2}{3}$, so the line does intersect the x-axis. The point of intersection is $\left(\frac{11}{6}, 0, 0\right)$

c Angle θ is such that

$$\cos\theta = \frac{\begin{pmatrix}2\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\left|\begin{pmatrix}2\\3\\-2\end{pmatrix}\right|\left|\begin{pmatrix}1\\0\\0\end{pmatrix}\right|} = \frac{2}{\sqrt{17}}$$

$\theta = 61.0°$

21 a The angle θ between the direction vectors $\mathbf{d_1} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$ and $\mathbf{d_2} = \begin{pmatrix}1\\5\\-3\end{pmatrix}$ satisfies

$$\cos\theta = \left|\frac{\mathbf{d_1}\cdot\mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|}\right| = \left|\frac{-11}{\sqrt{9}\sqrt{35}}\right|$$

$\theta = 51.7°$

b Solving for the intersection:

$$\begin{cases} 5 + 2\lambda = \mu & (1) \\ -3 - 2\lambda = 7 - 6\mu & (2) \\ 1 + \lambda = -5 + 4\mu & (3) \end{cases}$$

$(1) + (2): 2 = 7 - 5\mu \Rightarrow \mu = 1$

$(1): \lambda = \dfrac{\mu - 5}{2} = -2$

Substituting into $(3): 1 + \lambda = -1 = -5 + 4\mu$

The three equations are consistent so the lines intersect, at $X(1, 1, -1)$

c When $\lambda = 2$, $\mathbf{r_1} = \begin{pmatrix}9\\-7\\3\end{pmatrix}$ so $Y(9, -7, 3)$ lies on L_2

d $X\hat{Y}Z = 90°, Y\hat{X}Z = 51.7°, XY = \left|\begin{pmatrix}8\\-8\\4\end{pmatrix}\right| = 12$

Area $XYZ = \dfrac{1}{2}(XY)(XY\tan 51.7°) = 91.2$

22 a $\mathbf{r}_A = \begin{pmatrix}7\\-3\\2\end{pmatrix} + t\begin{pmatrix}-1\\0\\2\end{pmatrix}$

$\mathbf{r}_B = \begin{pmatrix}1\\1\\26\end{pmatrix} + t\begin{pmatrix}0\\1\\3\end{pmatrix}$

b i Re-expressing the two particle positions with different time parameters λ and μ respectively, and seeking an intersection of the two lines:

$$\begin{cases} 7 - \lambda = 1 & (1) \\ -3 = 1 + \mu & (2) \\ 2 + 2\lambda = 26 + 3\mu & (3) \end{cases}$$

$(1): \lambda = 6$

$(2): \mu = -4$

Substituting into (3): $2 + 2\lambda = 14 = 26 + 3\mu$

Since the system is consistent, the two paths do intersect.

ii $\mathbf{r}_A(6) = \mathbf{r}_B(-4) = \begin{pmatrix} 1 \\ -3 \\ 14 \end{pmatrix}$ so the paths intersect at $(1, -3, 14)$

c Since the two particles are at this intersection points at different times, they do not collide.

23 a The vector of travel is $\begin{pmatrix} 5 \\ -5 \end{pmatrix} - \begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

The angle of this direction clockwise from **i** (east) is $\tan^{-1}\left(-\frac{2}{6}\right) = -18.4°$

The bearing is therefore $090° + 18.4° = 108°$.

b Assuming a constant velocity, over 40 minutes the boat travelled $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$ so in kilometres per hour, the velocity is $\begin{pmatrix} 9 \\ -3 \end{pmatrix}$.

In kilometres, the position vector is $\mathbf{r} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} + t\begin{pmatrix} 9 \\ -3 \end{pmatrix}$.

c $0.5\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} + 2.5\begin{pmatrix} 9 \\ -3 \end{pmatrix}$

$0.5p = 21.5 \Rightarrow p = 43$

$0.5q = -10.5 \Rightarrow q = -21$

24 **a** $\mathbf{r} = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Displacement at $t = 5$ is $5\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ -5 \\ 10 \end{pmatrix}$

b Speed is $\left|\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\right| = \sqrt{14} \approx 3.74 \text{ m s}^{-1}$

c The given line has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}$

Solving for an intersection:

$$\begin{cases} 3 - 3\lambda = 4 + 3t & (1) \\ -\lambda = 6 - t & (2) \\ 1 + 4\lambda = -7 + 2t & (3) \end{cases}$$

$(1) - 3(2) \Rightarrow 3 = -14 + 6t \Rightarrow t = \frac{17}{6}$

$(2): \lambda = t - 6 \Rightarrow \lambda = -\frac{19}{6}$

Substituting into (3): $1 + 4\lambda = -\frac{35}{3} \neq -7 + 2t$

The system is inconsistent so the particle's path does not cross that line.

25 a $\mathbf{r_1} = \begin{pmatrix} 3t \\ 5-4t \\ t \end{pmatrix}$

b $\mathbf{r_2} = \begin{pmatrix} 5t \\ 2t \\ 7-t \end{pmatrix}$

$$\begin{aligned} d &= |\mathbf{r_1} - \mathbf{r_2}| \\ &= \left| \begin{pmatrix} -2t \\ 5-6t \\ 2t-7 \end{pmatrix} \right| \end{aligned}$$

$$\begin{aligned} d^2 &= 4t^2 + (5-6t)^2 + (2t-7)^2 \\ &= 44t^2 - 88t + 74 \end{aligned}$$

c $d^2 = 44(t-1)^2 + 30 \geq 30$

The planes will never be closer than $\sqrt{30}$ from each other so cannot collide.

d $\sqrt{30} = 5.48\,\text{km}$

26 a

$$\begin{aligned} |\mathbf{a} - \mathbf{b}| &= \sqrt{(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})} \\ &= \sqrt{\mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} - 2\mathbf{a}\cdot\mathbf{b}} \\ &= \sqrt{2 - 2\cos\alpha} \end{aligned}$$

$$\begin{aligned} |\mathbf{a} + \mathbf{b}| &= \sqrt{(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})} \\ &= \sqrt{\mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + 2\mathbf{a}\cdot\mathbf{b}} \\ &= \sqrt{2 + 2\cos\alpha} \end{aligned}$$

b $\sqrt{2 + 2\cos\alpha} = 4\sqrt{2 - 2\cos\alpha}$

$2 + 2\cos\alpha = 16(2 - 2\cos\alpha)$

$34\cos\alpha = 30$

$\cos\alpha = \frac{15}{17}$

$\alpha = 28.1°$

27 $\overrightarrow{PC} = \begin{pmatrix} -3+2\lambda \\ -\lambda \\ -4+2\lambda \end{pmatrix}$

Require $\overrightarrow{PC} \cdot \mathbf{d} = 0$ where $\mathbf{d} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is the direction vector of l.

$-14 + 9\lambda = 0 \Rightarrow \lambda = \frac{14}{9}$

C has coordinates $\left(\frac{64}{9}, \frac{4}{9}, \frac{19}{9}\right)$

28 a $l_2: \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$

b Both lines have direction $\mathbf{d} = \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu\mathbf{d} - \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} + \mu\mathbf{d}$$

Require $\overrightarrow{AB} \cdot \mathbf{d} = 0$

$$\mu = \frac{-1}{|\mathbf{d}|^2}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \cdot \mathbf{d} = -\frac{2}{34} = -\frac{1}{17}$$

B has coordinates $\left(\frac{30}{17}, -\frac{14}{17}, -\frac{3}{17}\right)$

c The separation between the lines is equal to AB

$$\left|\overrightarrow{AB}\right| = \left|\frac{1}{17}\begin{pmatrix} 30 \\ 3 \\ -37 \end{pmatrix}\right| = \frac{1}{17}\sqrt{2278} = 2.81$$

29 a $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2\lambda - 1 \\ \lambda + 21 \\ -3 \end{pmatrix}, \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} -7 \\ 4 \\ 2 \end{pmatrix}, \overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} -6 - 2\lambda \\ -17 - \lambda \\ 5 \end{pmatrix}$

If $BC \perp AC$ then $\overrightarrow{BC} \cdot \overrightarrow{AC} = 0$

$$42 + 14\lambda - 68 - 4\lambda + 10 = 0$$

$$10\lambda = 16$$

$$\lambda = 1.6$$

b $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2.2 \\ 22.6 \\ -3 \end{pmatrix}, \overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} -7 \\ 4 \\ 2 \end{pmatrix}, \overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{pmatrix} -9.2 \\ -18.6 \\ 5 \end{pmatrix}$

$$AB = \sqrt{2.2^2 + 22.6^2 + (-3)^2} = \sqrt{524.6}$$

$$AC = \sqrt{69}$$

$$BC = \sqrt{455.6}$$

$$\angle C = 90°$$

$$\angle A = \cos^{-1}\left(\frac{AC}{AB}\right) = 68.7°$$

$$\angle B = 90 - \angle A = 21.3°$$

c area $= \frac{1}{2}(AC)(BC) = 88.7$

30 $\mathbf{a} \times \mathbf{b} = k\mathbf{c}$

$$\begin{pmatrix} -5 \\ -1 - 3p \\ 15 \end{pmatrix} = k\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$$k = -5$$

$$-1 - 3p = -20$$

$$p = \frac{19}{3}$$

31 $A(2,1,2), B(-1,2,2)$ and $C(0,1,5)$

Area of a triangle with side vectors **u** and **v** is $\frac{1}{2}|\mathbf{u}\times\mathbf{v}|$

$\overrightarrow{AB} = \begin{pmatrix}-3\\1\\0\end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix}-2\\0\\3\end{pmatrix}$

$$\begin{aligned}\text{Area} &= \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|\\ &= \frac{1}{2}\left|\begin{pmatrix}3\\9\\2\end{pmatrix}\right|\\ &= \frac{1}{2}\sqrt{94} \approx 4.85\end{aligned}$$

32 $\mathbf{a} = \begin{pmatrix}1\\-1\\3\end{pmatrix}, \mathbf{b} = \begin{pmatrix}2q\\1\\q\end{pmatrix}$

Require $p\mathbf{a}+\mathbf{b} = k\begin{pmatrix}1\\1\\2\end{pmatrix}$

$$\begin{cases}p+2q=k & (1)\\ -p+1=k & (2)\\ 3p+q=2k & (3)\end{cases}$$

Eliminating k:

$$\begin{cases}(1)-(2): & 2p+2q-1=0 & (4)\\ (3)-2(2): & 5p+q-2=0 & (5)\end{cases}$$

$2(5)-(4): 8p-3=0 \Rightarrow p=\frac{3}{8}$

$(5): q = 2-5p = \frac{1}{8}$

33 The displacement from the top of the tree is $\begin{pmatrix}2+t\\1-t\\1+t\end{pmatrix} - \begin{pmatrix}1\\1\\3\end{pmatrix} = \begin{pmatrix}1+t\\-t\\t-2\end{pmatrix}$

The distance is therefore $d = \sqrt{(1+t)^2+(-t)^2+(t-2)^2}$

$$\begin{aligned}d^2 &= 3t^2-2t+5\\ &= 3\left(t^2-\frac{2}{3}t\right)+5\\ &= 3\left(\left(t-\frac{1}{3}\right)^2-\frac{1}{9}\right)+5\\ &= 3\left(t-\frac{1}{3}\right)^2+\frac{14}{3}\end{aligned}$$

The minimum value of d^2 is $\frac{14}{3}$ at $t=\frac{1}{3}$

The minimum distance is therefore $\sqrt{\frac{14}{3}}$

34 a

$$\begin{cases} -5 - 3\lambda = 3 + \mu & (1) \\ 1 = \mu & (2) \\ 10 + 4\lambda = -9 + 7\mu & (3) \end{cases}$$

$(2): \mu = 1$

$(1): \lambda = \dfrac{8 + \mu}{-3} = -3$

Substituting into (3):

$10 + 4\lambda = -2 = -9 + 7\mu$

The system is consistent so the two lines intersect. The point of intersection is $P(4, 1, -2)$

b When $\mu = 2$, $\mathbf{r_2} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$ so Q lies on the line.

c $\overrightarrow{QM} = \begin{pmatrix} -10 - 3\lambda \\ -1 \\ 5 + 4\lambda \end{pmatrix}$. Require $\overrightarrow{QM} \cdot \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = 0$

$30 + 9\lambda + 20 + 16\lambda = 0$

$50 = -25\lambda$

$\lambda = -2$

M has coordinates $(1, 1, 2)$

d $P\widehat{M}Q = 90°, PM = \left| \begin{matrix} -3 \\ 0 \\ 4 \end{matrix} \right| = 5, QM = \left| \begin{matrix} 4 \\ 1 \\ 3 \end{matrix} \right| = \sqrt{26}$

Therefore the area of triangle PQM is $\dfrac{1}{2} \times 5 \times \sqrt{26} = \dfrac{5\sqrt{26}}{2}$

35 a When $\lambda = \frac{5}{6}, \mathbf{r}_1 = \frac{1}{6}\begin{pmatrix} 5 \\ 19 \\ 27 \end{pmatrix}$ so P lies on l_1

When $t = \dfrac{7}{6}, \mathbf{r}_2 = \dfrac{1}{6}\begin{pmatrix} 5 \\ 19 \\ 27 \end{pmatrix}$ so P also lies on l_2

b The direction vectors of the lines are $\mathbf{d_1} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \mathbf{d_2} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$

The acute angle θ between the lines has

$\cos\theta = \left| \dfrac{\mathbf{d_1} \cdot \mathbf{d_2}}{|\mathbf{d_1}||\mathbf{d_2}|} \right| = \dfrac{|13|}{\sqrt{35}\sqrt{11}}$

$\theta = 48.5°$

c When $t = 3, \mathbf{r}_2 = \begin{pmatrix} -1 \\ 3 \\ 10 \end{pmatrix}$ so Q lies on l_2

d $\overrightarrow{PQ} = \frac{11}{6}\mathbf{d_2}$ so $PQ = \frac{11}{6}\sqrt{11} \approx 6.08$

e If the point on l_1 closest to Q is R then $Q\hat{R}P = 90°$, so $QR = PQ\sin\theta \approx 4.55$

36 a for $\mu = 3, \mathbf{r_2} = \begin{pmatrix} 8 \\ 2 \\ 6 \end{pmatrix}$ so $Q(8, 2, 6)$ lies on l_2

b Both lines pass through $P(2, -1, 0)$ and both direction vectors $\mathbf{d_1} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and

$\mathbf{d_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ have length

Then $PQ = 3|\mathbf{d_2}| = 9$

So $PR = 9$.

$\overrightarrow{PR} = \pm 3\mathbf{d_1} = \pm\begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$

R has coordinates $(5, -7, 6)$ or $(-1, 5, -6)$

c The angle bisector will have direction vector which is the sum of the two line direction vectors.

Since R could be on either side of P, the relevant direction vector for l_1 could be $\pm\mathbf{d_1}$.

$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ or $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$

37 a $\overrightarrow{PC} = \begin{pmatrix} -3 + 2\lambda \\ -\lambda \\ -4 + 2\lambda \end{pmatrix}$

Require $\overrightarrow{PC} \cdot \mathbf{d} = 0$ where $\mathbf{d} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ is the direction vector of l.

$-14 + 9\lambda = 0 \Rightarrow \lambda = \frac{14}{9}$

C has coordinates $\left(\frac{64}{9}, \frac{4}{9}, \frac{19}{9}\right)$

b $PC = \left|\frac{1}{9}\begin{pmatrix} 1 \\ -14 \\ -8 \end{pmatrix}\right| = \frac{\sqrt{29}}{3}$

c $\overrightarrow{PC} = \frac{1}{9}\begin{pmatrix} 1 \\ -14 \\ -8 \end{pmatrix} = \overrightarrow{CQ}$

Q has coordinates $\left(\frac{65}{9}, -\frac{10}{9}, \frac{11}{9}\right)$

38 a

$$\overrightarrow{PQ} = \left(\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right) - \left(\begin{pmatrix} 1 \\ -10 \\ 12 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}\right) = \begin{pmatrix} 3 - \lambda + \mu \\ 11 - 3\lambda \\ -7 + 4\lambda + 2\mu \end{pmatrix}$$

Require $\overrightarrow{PQ}$ is perpendicular to both lines, so $\overrightarrow{PQ} \cdot \mathbf{d_1} = \overrightarrow{PQ} \cdot \mathbf{d_2} = 0$

$\overrightarrow{PQ} \cdot \mathbf{d_1} = 0$:

$$\begin{pmatrix} 3-\lambda+\mu \\ 11-3\lambda \\ -7+4\lambda+2\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} = 64 - 26\lambda - 7\mu = 0$$

$\Rightarrow 26\lambda + 7\mu = 64 \qquad (1)$

$\overrightarrow{PQ} \cdot \mathbf{d_2} = 0$:

$$\begin{pmatrix} 3-\lambda+\mu \\ 11-3\lambda \\ -7+4\lambda+2\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -11 + 7\lambda + 5\mu = 0$$

$\Rightarrow 7\lambda + 5\mu = 11 \qquad (2)$

b $5(1) - 7(2): 81\lambda = 243$

$\lambda = 3$

$\Rightarrow \mu = -2$

$$\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$PQ = \sqrt{9} = 3$ is the shortest distance between the two lines.

39 a Area $BCD = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

b $h = |\mathbf{c}| \cos\theta$

c $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a} \times \mathbf{b}||\mathbf{c}| \cos\theta$

(Since θ is the acute angle between AE and AC, the value must be positive, but $\mathbf{a} \times \mathbf{b}$ may yield the upward of downward vector, depending on the orientation of the triangle, so an absolute value is needed in this formula)

$$\begin{aligned} \text{The volume of a tetrahedron} &= \frac{1}{3} \times \text{base area} \times \text{height} \\ &= \frac{1}{3}\left(\frac{1}{2}|\mathbf{a} \times \mathbf{b}|\right)|\mathbf{c}| \cos\theta \\ &= \frac{1}{6}|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| \end{aligned}$$

d $\mathbf{a} = \overrightarrow{CB} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{b} = \overrightarrow{CD} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}, \mathbf{c} = \overrightarrow{CA} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

$$\begin{aligned} \frac{1}{6}|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| &= \frac{1}{6}\left|\left(\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}\right| \\ &= \frac{1}{6}\left|\begin{pmatrix} -7 \\ -1 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}\right| \\ &= \frac{1}{6}|2| \\ &= \frac{1}{3} \end{aligned}$$

e

$$\text{Area } BCD = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$$
$$= \frac{1}{2}\left|\begin{pmatrix}-7\\-1\\7\end{pmatrix}\right|$$
$$= \frac{1}{2}\sqrt{99}$$

$$V = \frac{1}{3}h\times\left(\frac{1}{2}\sqrt{99}\right) = \frac{1}{3}$$

$$h = \frac{2}{\sqrt{99}} = \frac{2\sqrt{11}}{3}$$

f

$$\text{Area } ACD = \frac{1}{2}|\mathbf{c}\times\mathbf{b}|$$
$$= \frac{1}{2}\left|\begin{pmatrix}-1\\-2\\-1\end{pmatrix}\times\begin{pmatrix}2\\-7\\1\end{pmatrix}\right|$$
$$= \frac{1}{2}\left|\begin{pmatrix}-9\\-1\\11\end{pmatrix}\right|$$
$$= \frac{1}{2}\sqrt{203}$$

Therefore the distance from B to the face ACD is $\frac{2\sqrt{203}}{203} < \frac{2\sqrt{11}}{3}$.

B is closer to its opposite face.

40 Let the meeting point be C

Require that the path of S_2, which is BC, should be of minimal length.

It follows that AC must be perpendicular to BC.

Since the velocity of S_1 is $\begin{pmatrix}10\\20\end{pmatrix}$, the velocity of S_2 must be $\pm\begin{pmatrix}-60\\30\end{pmatrix}$, if it is to be perpendicular and with a magnitude three times as great.

BC has equation $\mathbf{r} = \begin{pmatrix}70\\30\end{pmatrix} + \lambda\begin{pmatrix}-60\\30\end{pmatrix}$, while AC has equation $\mathbf{r} = t\begin{pmatrix}10\\20\end{pmatrix}$

These intersect at C:

$$\begin{cases}70 - 60\lambda = 10t & (1)\\ 30 + 30\lambda = 20t & (2)\end{cases}$$

$2(2) + (1): 130 = 50t \Rightarrow t = 2.6$

$$\lambda = \frac{70 - 10t}{60} = \frac{44}{60}$$

The second ship requires 44 minutes to reach the intersection, and the first ship takes 2 hours and 36 minutes.

If the first ship leaves at $10:00$, they meet at $12:36$ so the second ship leaves port at $11:52$.

3 Matrices

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 3A

23 The matrix for overall results is given by

$$\mathbf{H}+\mathbf{A}=\begin{pmatrix}5&2&3\\4&2&4\\6&3&1\end{pmatrix}+\begin{pmatrix}4&1&5\\4&0&6\\5&2&3\end{pmatrix}=\begin{pmatrix}9&3&8\\8&2&10\\11&5&4\end{pmatrix}$$

24 The matrix for next week's projected takings is given by

$$1.08\begin{pmatrix}800&560\\640&320\\680&300\end{pmatrix}=\begin{pmatrix}864.00&604.80\\691.20&345.60\\734.40&324.00\end{pmatrix}$$

25 $\begin{pmatrix}2x-y&3\\7&y+3x\end{pmatrix}=\begin{pmatrix}12&3\\7&13\end{pmatrix}$

So

$$\begin{cases}2x-y=12\\y+3x=13\end{cases}$$

Solving simultaneously: $x=5, y=-$

26

$$\begin{pmatrix}1&2p&0\\p&-2&3q\end{pmatrix}=\begin{pmatrix}1&5-q&0\\p&-2&p-13\end{pmatrix}$$

So

$$\begin{cases}2p=5-q\\3q=p-13\end{cases}$$

$$\begin{cases}2p+q=5\\p-3q=13\end{cases}$$

Solving simultaneously: $p=4, q=-3$

27 a

$$\begin{aligned}\mathbf{A}+3\mathbf{B}&=\begin{pmatrix}-3&0\\5&4\end{pmatrix}+3\begin{pmatrix}-1&5\\k&-k\end{pmatrix}\\&=\begin{pmatrix}-3&0\\5&4\end{pmatrix}+\begin{pmatrix}-3&15\\3k&-3k\end{pmatrix}\\&=\begin{pmatrix}-6&15\\5+3k&4-3k\end{pmatrix}\end{aligned}$$

b

$$\begin{aligned}2\mathbf{A}-\mathbf{B}+4\mathbf{I} &= 2\begin{pmatrix}-3 & 0\\ 5 & 4\end{pmatrix}-\begin{pmatrix}-1 & 5\\ k & -k\end{pmatrix}+4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}-6 & 0\\ 10 & 8\end{pmatrix}-\begin{pmatrix}-1 & 5\\ k & -k\end{pmatrix}+\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}\\ &= \begin{pmatrix}-1 & -5\\ 10-k & 12+k\end{pmatrix}\end{aligned}$$

28 a

$$\begin{aligned}\mathbf{A}-\mathbf{B} &= \begin{pmatrix}1 & k\\ 2 & 3\end{pmatrix}-\begin{pmatrix}-k & 2k\\ 1 & 3\end{pmatrix}\\ &= \begin{pmatrix}1+k & -k\\ 1 & 0\end{pmatrix}\end{aligned}$$

b

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix}1 & k\\ 2 & 3\end{pmatrix}\begin{pmatrix}-k & 2k\\ 1 & 3\end{pmatrix}\\ &= \begin{pmatrix}-k+k & 2k+3k\\ -2k+3 & 4k+9\end{pmatrix}\\ &= \begin{pmatrix}0 & 5k\\ 3-2k & 4k+9\end{pmatrix}\end{aligned}$$

29 a

$$\begin{aligned}\mathbf{P}+\mathbf{Q} &= \begin{pmatrix}1 & k & -3\\ 2 & 4 & k\\ 1 & 0 & 5\end{pmatrix}+\begin{pmatrix}6 & 1 & 4\\ -1 & k & 3\\ 2 & 0 & k\end{pmatrix}\\ &= \begin{pmatrix}7 & k+1 & 1\\ 1 & k+4 & k+3\\ 3 & 0 & k+5\end{pmatrix}\end{aligned}$$

b

$$\begin{aligned}\mathbf{PQ} &= \begin{pmatrix}1 & k & -3\\ 2 & 4 & k\\ 1 & 0 & 5\end{pmatrix}\begin{pmatrix}6 & 1 & 4\\ -1 & k & 3\\ 2 & 0 & k\end{pmatrix}\\ &= \begin{pmatrix}6-k-6 & 1+k^2+0 & 4+3k-3k\\ 12-4+2k & 2+4k+0 & 8+12+k^2\\ 6+0+10 & 1+0+0 & 4+0+5k\end{pmatrix}\\ &= \begin{pmatrix}-k & k^2+1 & 4\\ 2k+8 & 4k+2 & k^2+20\\ 16 & 1 & 5k+4\end{pmatrix}\end{aligned}$$

30 a

$$\mathbf{C} = \begin{pmatrix}50 & 38 & 24 & 80\\ 55 & 30 & 26 & 75\end{pmatrix}\begin{pmatrix}100\\ 50\\ 125\\ 30\end{pmatrix}$$

$$= \begin{pmatrix}12300\\ 12500\end{pmatrix}$$

b Cost from France = £12 300

Cost from Germany = £12 500

31

$$\begin{pmatrix} x & y \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ 5 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 3x + y & x + 2y \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ 5 & 0 \end{pmatrix}$$

So

$$\begin{cases} 3x + y = 7 \\ x + 2y = 9 \end{cases}$$

Solving simultaneously: $x = 1, y = 4$

32 $\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & p \\ 3 & q \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 11 & 11 \end{pmatrix}$

$$\begin{pmatrix} 5 & 2p + q \\ 11 & 4q - p \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 11 & 11 \end{pmatrix}$$

So

$$\begin{cases} 2p + q = -4 \\ 4q - p = 11 \end{cases}$$

Solving simultaneously: $p = -3, q = 2$

33 a

$$\mathbf{M} = \begin{pmatrix} 210 & 180 & 320 & 400 \\ 250 & 150 & 200 & 450 \end{pmatrix}\begin{pmatrix} 55 & 50 \\ 80 & 85 \\ 48 & 40 \\ 20 & 22 \end{pmatrix}$$
$$= \begin{pmatrix} 49310 & 47400 \\ 44350 & 43150 \end{pmatrix}$$

b Engineer Right should use factory H as total cost is £47 400 as opposed to £49 310 at factory G

Forge Well should also use factory H as total cost is £43 150 as opposed to £44 350 at factory G

34 a $\begin{pmatrix} 0.28 & 0.34 \\ 0.42 & 0.31 \\ 0.30 & 0.35 \end{pmatrix}\begin{pmatrix} 26124 & 28987 \\ 30125 & 29846 \end{pmatrix} = \begin{pmatrix} 17557 & 18264 \\ 20311 & 21427 \\ 18381 & 19142 \end{pmatrix}$

b Projected total for Yellow Part = 18 381 + 19 142 = 37 523

35

$$\mathbf{A} + s\mathbf{B} = t\mathbf{I}$$
$$\begin{pmatrix} a & 3 \\ -2 & 1 \end{pmatrix} + s\begin{pmatrix} 2 & b \\ 1 & 3 \end{pmatrix} = t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} a + 2s & 3 + bs \\ -2 + s & 1 + 3s \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

So

$$\begin{cases} a + 2s = t & (1) \\ 3 + bs = 0 & (2) \\ -2 + s = 0 & (3) \\ 1 + 3s = t & (4) \end{cases}$$

From (3): $s = 2$

From (4): $1 + 6 = t \Rightarrow t = 7$

From (1): $a + 4 = 7 \Rightarrow a = 3$

From (2): $3 + 2b = 0 \Rightarrow b = -1.5$

36

$$\frac{1}{2}\boldsymbol{C} = p\boldsymbol{A} + q\boldsymbol{B}$$
$$\frac{1}{2}\begin{pmatrix} 2 & 2 \\ 4 & 0 \end{pmatrix} = p\begin{pmatrix} 1 & a \\ 2 & 2 \end{pmatrix} + q\begin{pmatrix} b & 3 \\ -4 & -5 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} p + bq & ap + 3q \\ 2p - 4q & 2p - 5q \end{pmatrix}$$

So

$$\begin{cases} 1 = p + bq & (1) \\ 1 = ap + 3q & (2) \\ 2 = 2p - 4q & (3) \\ 0 = 2p - 5q & (4) \end{cases}$$

Solving (3) and (4) simultaneously: $p = 5, q = 2$

From (1): $1 = 5 + 2b \Rightarrow b = -2$

From (2): $1 = 5a + 6 \Rightarrow a = -1$

37

$$c\mathbf{M} + d\mathbf{N} = \mathbf{I}$$
$$c\begin{pmatrix} 1 & 2a \\ -a & 3 \end{pmatrix} + d\begin{pmatrix} 4 & b+1 \\ 3b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} c + 4d & 2ac + bd + d \\ -ac + 3bd & 3c + d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$\begin{cases} c + 4d = 1 & (1) \\ 2ac + bd + d = 0 & (2) \\ -ac + 3bd = 0 & (3) \\ 3c + d = 1 & (4) \end{cases}$$

Solving (1) and (4) simultaneously: $c = \frac{3}{11}, d = \frac{2}{11}$

Substituting into (2) and (3):

$$\begin{cases} \dfrac{6}{11}a + \dfrac{2}{11}b = -\dfrac{2}{11} \\ -\dfrac{3}{11}a + \dfrac{6}{11}b = 0 \end{cases}$$

Solving simultaneously: $a = -\frac{2}{7}, b = -\frac{1}{7}$

38 Since the two matrices commute,

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} k & 6 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} k & 6 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$
$$\begin{pmatrix} k + 12 & 3 \\ 2k + 16 & 8 \end{pmatrix} = \begin{pmatrix} k + 12 & 3k + 24 \\ 2 & 8 \end{pmatrix}$$

$3 = 3k + 24 \Rightarrow k = -7$

Check in $2k + 16 = 2$: $-14 + 16 = 2$

So consistent. Therefore $k = -7$.

Note: It is possible that there is no value for k for which these matrices commute so it is important to check that the value of k found in the first equation is consistent in any other equations.

39 Since the two matrices commute,

$$\begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} c & d \\ d & -c \end{pmatrix} = \begin{pmatrix} c & d \\ d & -c \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 5 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 2c + 5d & 2d - 5c \\ 5c - 2d & 5d - 2c \end{pmatrix} = \begin{pmatrix} 2c + 5d & 5c - 2d \\ 2d - 5c & 5d - 2c \end{pmatrix}$$

$$\begin{aligned} 2d - 5c &= 5c - 2d \\ 4d &= 10c \\ d &= 2.5c \end{aligned}$$

40 a

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{BC} &= \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \\ &= \begin{pmatrix} b_1c_1 + b_2c_3 & b_1c_2 + b_2c_4 \\ b_3c_1 + b_4c_3 & b_3c_2 + b_4c_4 \end{pmatrix} \end{aligned}$$

b

$$(\mathbf{AB})\mathbf{C} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1c_1 + a_2b_3c_1 + a_1b_2c_3 + a_2b_4c_3 & a_1b_1c_2 + a_2b_3c_2 + a_1b_2c_4 + a_2b_4c_4 \\ a_3b_1c_1 + a_4b_3c_1 + a_3b_2c_3 + a_4b_4c_3 & a_3b_1c_2 + a_4b_3c_2 + a_3b_2c_4 + a_4b_4c_4 \end{pmatrix}$$

$$\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1c_1 + b_2c_3 & b_1c_2 + b_2c_4 \\ b_3c_1 + b_4c_3 & b_3c_2 + b_4c_4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1b_1c_1 + a_1b_2c_3 + a_2b_3c_1 + a_2b_4c_3 & a_1b_1c_2 + a_1b_2c_4 + a_2b_3c_2 + a_2b_4c_4 \\ a_3b_1c_1 + a_3b_2c_3 + a_4b_3c_1 + a_4b_4c_3 & a_3b_1c_2 + a_3b_2c_4 + a_4b_3c_2 + a_4b_4c_4 \end{pmatrix}$$

So $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$

Exercise 3B

26 a

$$\begin{aligned} \det \mathbf{A} &= 0 \\ 2k^2 - 4 &= 0 \\ k^2 &= 2 \\ k &= \pm\sqrt{2} \end{aligned}$$

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{\det \mathbf{A}}\begin{pmatrix} k & -1 \\ -4 & 2k \end{pmatrix} \\ &= \frac{1}{2k^2 - 4}\begin{pmatrix} k & -1 \\ -4 & 2k \end{pmatrix} \end{aligned}$$

27 a

$$\det \mathbf{A} = 2 — 3c^2$$
$$= 2 + 3c^2$$
$$> 0 \quad \text{for all real } c$$

Since $\det \mathbf{A} \neq 0$, $\mathbf{A}$ has an inverse for all real c

b

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{pmatrix} 2 & -3c \\ c & 1 \end{pmatrix}$$
$$= \frac{1}{3c^2+2}\begin{pmatrix} 2 & -3c \\ c & 1 \end{pmatrix}$$

28 a

$$\mathbf{A}^{-1} = \frac{1}{-16+12}\begin{pmatrix} -2 & -4 \\ 3 & 8 \end{pmatrix}$$
$$= \begin{pmatrix} 0.5 & 1 \\ -0.75 & -2 \end{pmatrix}$$

b

$$\mathbf{AB} = \mathbf{C}$$
$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$
$$= \begin{pmatrix} 0.5 & 1 \\ -0.75 & -2 \end{pmatrix}\begin{pmatrix} 28 & 20 & 40 \\ -13 & -7 & -18 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 3 & 2 \\ 5 & -1 & 6 \end{pmatrix}$$

29 a From the GDC

$$\mathbf{B}^{-1} = \frac{1}{117}\begin{pmatrix} -7 & 19 & 16 \\ 19 & 32 & -10 \\ 25 & -1 & -7 \end{pmatrix}$$

b

$$\mathbf{AB} = \mathbf{C}$$
$$\mathbf{A} = \mathbf{CB}^{-1}$$
$$= \begin{pmatrix} -9 & 19 & -31 \\ 25 & -10 & 38 \end{pmatrix}\frac{1}{117}\begin{pmatrix} -7 & 19 & 16 \\ 19 & 32 & -10 \\ 25 & -1 & -7 \end{pmatrix}$$
$$= \begin{pmatrix} -3 & 4 & -1 \\ 5 & 1 & 2 \end{pmatrix}$$

30 Let the unknown 3×2 matrix be $\mathbf{M}$.

Then

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{pmatrix}\mathbf{M} = \begin{pmatrix} 1 & -16 \\ 23 & 20 \\ 35 & 21 \end{pmatrix}$$
$$\mathbf{M} = \begin{pmatrix} 1 & -16 \\ 23 & 20 \\ 35 & 21 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 7 & 2 \\ 4 & 4 \\ 3 & 9 \end{pmatrix}$$

31

$$\mathbf{P}^{-1}\mathbf{QP} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$
$$\mathbf{QP} = \mathbf{P}\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$
$$\mathbf{Q} = \mathbf{P}\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}\mathbf{P}^{-1}$$
$$= \begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}\frac{1}{8}\begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1.75 & 2.25 \\ 3.75 & 0.25 \end{pmatrix}$$

32

$$\mathbf{ABC} = \mathbf{I}$$
$$\mathbf{BC} = \mathbf{A}^{-1}$$
$$\mathbf{C} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$
$$= \begin{pmatrix} 6 & 2 & -2 \\ -3 & -1 & 2 \\ -5 & -1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 5 & 1 & -4 \\ 3 & 0 & 2 \\ -1 & -1 & 6 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} -4.25 & 5.5 & -4.75 \\ 3.75 & -4.5 & 4.25 \\ -9.5 & 12.5 & -10.5 \end{pmatrix}$$

33

$$\det(k\mathbf{M}) = \det\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$
$$= k^2ad - k^2bc$$
$$= k^2(ad - bc)$$
$$= k^2 \det \mathbf{M}$$

34 a $\mathbf{M}^3 = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}^3 = \begin{pmatrix} 1 & 7 & 38 \\ 0 & 8 & -28 \\ 0 & 0 & 64 \end{pmatrix}$

$$7\mathbf{M}^2 - 14\mathbf{M} + 8\mathbf{I} = 7\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}^2 - 14\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 7 & 21 & 63 \\ 0 & 28 & -42 \\ 0 & 0 & 112 \end{pmatrix} - \begin{pmatrix} 14 & 14 & 28 \\ 0 & 28 & -14 \\ 0 & 0 & 56 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 7 & 38 \\ 0 & 8 & -28 \\ 0 & 0 & 64 \end{pmatrix}$$

So $\mathbf{M}^3 = 7\mathbf{M}^2 - 14\mathbf{M} + 8\mathbf{I}$

b

$$\mathbf{M}^3 = 7\mathbf{M}^2 - 14\mathbf{M} + 8\mathbf{I}$$
$$\mathbf{M}^2 = 7\mathbf{M} - 14\mathbf{I} + 8\mathbf{M}^{-1} \quad \text{(multiplying through by } \mathbf{M}^{-1}\text{)}$$
$$8\mathbf{M}^{-1} = \mathbf{M}^2 - 7\mathbf{M} + 14\mathbf{I}$$
$$\mathbf{M}^{-1} = \frac{1}{8}(\mathbf{M}^2 - 7\mathbf{M} + 14\mathbf{I})$$

35 Let $\mathbf{M}^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

Then, since $\mathbf{MM}^{-1} = \mathbf{I}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$\begin{cases} ae+bg=1 & (1) \\ af+bh=0 & (2) \\ ce+dg=0 & (3) \\ cf+dh=1 & (4) \end{cases}$$

From (3): $g = -\frac{ce}{d}$ (5)

Substituting (5) into (1):

$$ae + b\left(-\frac{ce}{d}\right) = 1$$
$$e(ad-bc) = d$$
$$e = \frac{d}{ad-bc}$$

Substituting into (5):

$$g = \frac{-c}{ad-bc}$$

From (2): $h = -\frac{af}{b}$ (6)

Substituting (6) into (4):

$$cf + d\left(-\frac{af}{b}\right) = 1$$
$$f(bc-ad) = b$$
$$f = \frac{-b}{ad-bc}$$

Substituting into (6):

$$h = \frac{a}{ad-bc}$$

So

$$\mathbf{M}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

36 a

$$\begin{aligned} (\mathbf{AB})(\mathbf{AB})^{-1} &= \mathbf{I} \\ \mathbf{A}^{-1}\mathbf{AB}(\mathbf{AB})^{-1} &= \mathbf{A}^{-1}\mathbf{I} && \text{(multiplying on the left by } \mathbf{A}^{-1}) \\ \mathbf{B}(\mathbf{AB})^{-1} &= \mathbf{A}^{-1} && \text{(since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}) \\ \mathbf{B}^{-1}\mathbf{B}(\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} && \text{(multipying on the left by } \mathbf{B}^{-1}) \\ (\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} && \text{(since } \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}) \end{aligned}$$

b

$$\begin{aligned}(\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{A}^{-1} &= \mathbf{B}^{-1}(\mathbf{A}^{-1})^{-1}\mathbf{A}^{-1} && \text{(by part a)}\\ &= \mathbf{B}^{-1}\mathbf{A}\mathbf{A}^{-1} && \text{(since } (\mathbf{A}^{-1})^{-1} = \mathbf{A})\\ &= \mathbf{B}^{-1}\end{aligned}$$

Exercise 3C

7 **a** $\begin{pmatrix}5 & -4\\3 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}7\\-1\end{pmatrix}$

where $\mathbf{A} = \begin{pmatrix}5 & -4\\3 & -2\end{pmatrix}$, $\mathbf{x} = \begin{pmatrix}x\\y\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}7\\-1\end{pmatrix}$

b

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{1}{-10-(-12)}\begin{pmatrix}-2 & 4\\-3 & 5\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}-2 & 4\\-3 & 5\end{pmatrix}\end{aligned}$$

c

$$\begin{aligned}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}\\ \begin{pmatrix}x\\y\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}-2 & 4\\-3 & 5\end{pmatrix}\begin{pmatrix}7\\-1\end{pmatrix}\\ &= \begin{pmatrix}-9\\-13\end{pmatrix}\end{aligned}$$

i.e. $x = -9, y = -13$

8 **a**

$$\begin{pmatrix}1 & -1 & 1\\5 & 3 & 2\\0 & 4 & -3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\4\end{pmatrix}$$

where $\mathbf{A} = \begin{pmatrix}1 & -1 & 1\\5 & 3 & 2\\0 & 4 & -3\end{pmatrix}$, $\mathbf{x} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\1\\4\end{pmatrix}$

b From the GDC,

$$\mathbf{A}^{-1} = -\frac{1}{12}\begin{pmatrix}-17 & 1 & -5\\15 & -3 & 3\\20 & -4 & 8\end{pmatrix}$$

c

$$\begin{aligned}\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}\\ &= -\frac{1}{12}\begin{pmatrix}-17 & 1 & -5\\15 & -3 & 3\\20 & -4 & 8\end{pmatrix}\begin{pmatrix}1\\1\\4\end{pmatrix}\\ &= \begin{pmatrix}3\\-2\\-4\end{pmatrix}\end{aligned}$$

i.e. $x = 3, y = -2, z = -4$

9 Let the quantity of raspberries be r kg and the quantity of strawberries be s kg.

Then

$$\begin{pmatrix} 8.5 & 5.5 \\ 8 & 6 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 26.5 \\ 27 \end{pmatrix}$$

So

$$\begin{aligned}\begin{pmatrix} r \\ s \end{pmatrix} &= \begin{pmatrix} 8.5 & 5.5 \\ 8 & 6 \end{pmatrix}^{-1}\begin{pmatrix} 26.5 \\ 27 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} 6 & -5.5 \\ -8 & 8.5 \end{pmatrix}\begin{pmatrix} 26.5 \\ 27 \end{pmatrix} \\ &= \begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix}\end{aligned}$$

i.e. the restaurant owner buys 1.5 kg or raspberries and 2.5 kg of strawberries.

10 a Let the amount of money invested in Allshare be $\$A$, the amount invested in Baserate be $\$B$ and the amount invested in Chartwise be $\$C$.

Total invested \$50 000, so $A + B + C = 50\,000$

\$10 000 more invested in A than C, so $A - C = 10\,000$

1 year returns of 5.2%, 3.1% and 6.5% giving a total of \$52 447.50, so

$$1.052A + 1.031B + 1.065C = 52447.5$$

Hence

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.052 & 1.031 & 1.065 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 50\,000 \\ 10\,000 \\ 52\,447.5 \end{pmatrix}$$

b

$$\begin{aligned}\begin{pmatrix} A \\ B \\ C \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.052 & 1.031 & 1.065 \end{pmatrix}^{-1}\begin{pmatrix} 50\,000 \\ 10\,000 \\ 52\,447.5 \end{pmatrix} \\ &= \begin{pmatrix} 22\,500 \\ 15\,000 \\ 12\,500 \end{pmatrix}\end{aligned}$$

i.e. amounts invested were, $A = \$22\,500, B = \$15\,000, C = \$12\,500$

11 a

$$\begin{pmatrix} k & 3 \\ k-2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

where $\mathbf{A} = \begin{pmatrix} k & 3 \\ k-2 & 4 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b Equations do not have a solution when $\mathbf{A}^{-1}$ doesn't exist, i.e. when $\det \mathbf{A} = 0$:

$$\begin{aligned}4k - 3(k-2) &= 0 \\ k + 6 &= 0 \\ k &= -6\end{aligned}$$

c

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{k+6}\begin{pmatrix} 4 & -3 \\ 2-k & k \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{k+6}\begin{pmatrix} -7 \\ 2k-2 \end{pmatrix}$$

i.e. $x = -\frac{7}{k+6}, y = \frac{2k-2}{k+6}$

12 **a** $\begin{pmatrix} k & 5 \\ 2 & k-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

where $\mathbf{A} = \begin{pmatrix} k & 5 \\ 2 & k-3 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

b Equations do not have a solution when $\mathbf{A}^{-1}$ doesn't exist, i.e. when $\det \mathbf{A} = 0$:

$$k(k-3) - 10 = 0$$
$$k^2 - 3k - 10 = 0$$
$$k = 5 \text{ or } -2$$

c

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{k^2-3k-10}\begin{pmatrix} k-3 & -5 \\ -2 & k \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{k^2-3k-10}\begin{pmatrix} 2k-1 \\ -k-4 \end{pmatrix}$$

i.e. $x = \frac{2k-1}{k^2-3k-10}, y = \frac{-k-4}{k^2-3k-10}$

Exercise 3D

10 **a** **i**

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$$
$$(7-\lambda)(4-\lambda) - 4 = 0$$
$$\lambda^2 - 11\lambda + 24 = 0$$

ii

$$(\lambda - 3)(\lambda - 8) = 0$$
$$\lambda = 3 \text{ or } 8$$

b $\underline{\lambda_1 = 3:}$

$$\begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$4x + 2y = 0 \Rightarrow y = -2x$

$v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\underline{\lambda_2 = 8:}$

$$\begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 8\begin{pmatrix} x \\ y \end{pmatrix}$$

$-x + 2y = 0 \Rightarrow x = 2y$

$v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

11 $\mathbf{M} = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= 0 \\ (-2-\lambda)(1-\lambda) - 4 &= 0 \\ \lambda^2 + \lambda - 6 &= 0 \\ (\lambda - 2)(\lambda + 3) &= 0 \\ \lambda &= 2 \text{ or } -3 \end{aligned}$$

$\lambda_1 = 2$:

$\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$

$-4x + 2y = 0 \Rightarrow 2x = y$

$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -3$:

$\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$

$x + 2y = 0 \Rightarrow x = -2y$

$v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

12 $\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= 0 \\ (4-\lambda)(-9-\lambda) + 30 &= 0 \\ \lambda^2 + 5\lambda - 6 &= 0 \\ (\lambda - 1)(\lambda + 6) &= 0 \\ \lambda &= 1 \text{ or } -6 \end{aligned}$$

$\underline{\lambda_1 = 1}$:

$\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$3x - 5y = 0 \Rightarrow 3x = 5y$

$v_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\underline{\lambda_2 = -6}$:

$\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -6\begin{pmatrix} x \\ y \end{pmatrix}$

$2x - y = 0 \Rightarrow y = 2x$

$v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

13 a $\begin{pmatrix} 3 & 1 \\ 4 & a \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{cases} 3 + 2 = \lambda & (1) \\ 4 + 2a = 2\lambda & (2) \end{cases}$$

From (1): $\lambda = 5$

Substituting into (2): $4 + 2a = 10 \Rightarrow a = 3$

b **i** From a), $\lambda_1 = 5$

ii $\det \mathbf{A} = \lambda_1 \lambda_2$

$9 - 4 = 5\lambda_2$
$\lambda_2 = 1$

$\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$

$2x + y = 0 \Rightarrow y = -2x$

$v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

14 **a**

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= 0 \\ (-5 - \lambda)(-7 - \lambda) - 24 &= 0 \\ \lambda^2 + 12\lambda + 11 &= 0 \\ (\lambda + 1)(\lambda + 11) &= 0 \\ \lambda &= -1 \text{ or } -11 \end{aligned}$$

$\underline{\lambda_1 = -1:}$

$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$

$-4x + 8y = 0 \Rightarrow x = 2y$

$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = -11:}$

$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -11 \begin{pmatrix} x \\ y \end{pmatrix}$

$6x + 8y = 0 \Rightarrow 3x = -4y$

$v_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

b $\mathbf{C} = \begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & -11 \end{pmatrix}$

15 **a**

$$\begin{aligned} \det(\mathbf{M} - \lambda \mathbf{I}) &= 0 \\ (8 - \lambda)(5 - \lambda) - (-2) &= 0 \\ \lambda^2 - 13\lambda + 42 &= 0 \\ (\lambda - 6)(\lambda - 7) &= 0 \\ \lambda &= 6 \text{ or } 7 \end{aligned}$$

$\underline{\lambda_1 = 6:}$

$\begin{pmatrix} 8 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix}$

$2x + 2y = 0 \Rightarrow x = -y$

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 7:}$

$\begin{pmatrix} 8 & 2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$

$x + 2y = 0 \Rightarrow x = -2y$

$v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

So, $\mathbf{P} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & 7 \end{pmatrix}$

b

$$\begin{aligned}
\mathbf{M}^3 &= \mathbf{PD}^3\mathbf{P}^{-1} \\
&= \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 7 \end{pmatrix}^3\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 6^3 & 0 \\ 0 & 7^3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -216 & -686 \\ 216 & 343 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 470 & 254 \\ -127 & 89 \end{pmatrix}
\end{aligned}$$

16 a

$$\begin{aligned}
\det(\mathbf{M} - \lambda\mathbf{I}) &= 0 \\
(7-\lambda)(1-\lambda) - (-8) &= 0 \\
\lambda^2 - 8\lambda + 15 &= 0 \\
(\lambda - 3)(\lambda - 5) &= 0 \\
\lambda &= 3 \text{ or } 5
\end{aligned}$$

$\underline{\lambda_1 = 3:}$

$\begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$

$4x + 4y = 0 \Rightarrow x = -y$

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 5:$

$\begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$

$2x + 4y = 0 \Rightarrow x = -2y$

$v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

So, $\mathbf{P} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

b

$$\begin{aligned}
\mathbf{M}^n &= \mathbf{PD}^n\mathbf{P}^{-1} \\
&= \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}^n\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}^{-1}
\end{aligned}$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3^n & 2(5^n) \\ -3^n & -5^n \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2(5^n) - 3^n & 2(5^n) - 2(3^n) \\ 3^n - 5^n & 2(3^n) - 5^n \end{pmatrix}$$

17 a

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\ (4-\lambda)(-4-\lambda) - (-12) &= 0 \\ \lambda^2 - 4 &= 0 \\ (\lambda - 2)(\lambda + 2) &= 0 \\ \lambda &= 2 \text{ or } -2 \end{aligned}$$

$\underline{\lambda_1 = 2:}$

$$\begin{pmatrix} 4 & -4 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

$2x - 4y = 0 \Rightarrow x = 2y$

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = -2:}$

$$\begin{pmatrix} 4 & -4 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -2 \begin{pmatrix} x \\ y \end{pmatrix}$$

$6x - 4y = 0 \Rightarrow 3x = 2y$

$$v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

So, $\mathbf{P} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

b

$$\begin{aligned} \mathbf{A}^n &= \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} \\ &= \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}^n \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & (-2)^n \end{pmatrix} \frac{1}{4} \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2(2^n) & 2(-2)^n \\ 2^n & 3(-2)^n \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 6(2^n) - 2(-2)^n & -4(2^n) + 4(-2)^n \\ 3(2^n) - 3(-2)^n & -2(2^n) + 6(-2)^n \end{pmatrix} \end{aligned}$$

i When n is odd $(-2)^n = -2^n$, so

$$\begin{aligned} \mathbf{A}^n &= \frac{1}{4} \begin{pmatrix} 6(2^n) + 2(2^n) & -4(2^n) - 4(2^n) \\ 3(2^n) + 3(2^n) & -2(2^n) - 6(2^n) \end{pmatrix} \\ &= \begin{pmatrix} 2(2^n) & -2(2^n) \\ 3(2^{n-1}) & -2(2^n) \end{pmatrix} \\ &= 2^{n-1} \begin{pmatrix} 4 & -4 \\ 3 & -4 \end{pmatrix} \end{aligned}$$

ii When n is even $(-2)^n = 2^n$

$$\begin{aligned}\mathbf{A}^n &= \frac{1}{4}\begin{pmatrix}6(2^n)-2(2^n) & -4(2^n)+4(2^n)\\ 3(2^n)-3(2^n) & -2(2^n)+6(2^n)\end{pmatrix}\\ &= \begin{pmatrix}2^n & 0\\ 0 & 2^n\end{pmatrix}\\ &= 2^n\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{aligned}$$

18 a For eigenvector $\begin{pmatrix}1\\3\end{pmatrix}$ with eigenvalue 1 and eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$ with eigenvalue – 2,

$$\begin{aligned}\mathbf{M}_1 &= \begin{pmatrix}1 & 1\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}1 & 1\\ 3 & 1\end{pmatrix}^{-1}\\ &= -\frac{1}{2}\begin{pmatrix}1 & -2\\ 3 & -2\end{pmatrix}\begin{pmatrix}1 & -1\\ -3 & 1\end{pmatrix}\\ &= \begin{pmatrix}-3.5 & 1.5\\ -4.5 & 2.5\end{pmatrix}\end{aligned}$$

For eigenvector $\begin{pmatrix}1\\3\end{pmatrix}$ with eigenvalue -2 and eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$ with eigenvalue 1,

$$\begin{aligned}\mathbf{M}_2 &= \begin{pmatrix}1 & 1\\ 3 & 1\end{pmatrix}\begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 3 & 1\end{pmatrix}^{-1}\\ &= -\frac{1}{2}\begin{pmatrix}-2 & 1\\ -6 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -3 & 1\end{pmatrix}\\ &= \begin{pmatrix}2.5 & -1.5\\ 4.5 & -3.5\end{pmatrix}\end{aligned}$$

b

$$\begin{aligned}\mathbf{M}_1\mathbf{M}_2 &= \begin{pmatrix}-3.5 & 1.5\\ -4.5 & 2.5\end{pmatrix}\begin{pmatrix}2.5 & -1.5\\ 4.5 & -3.5\end{pmatrix}\\ &= \begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}\\ &= -2\mathbf{I}\end{aligned}$$

So

$$\begin{aligned}\mathbf{M}_1\mathbf{M}_2\mathbf{v} &= -2\mathbf{I}\mathbf{v}\\ (\mathbf{M}_1\mathbf{M}_2)\mathbf{v} &= (-2\times 1)\mathbf{v}\end{aligned}$$

Therefore any eigenvector of the matrix $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2$ will have eigenvector -2, which is the product of the original eigenvectors.

19 a

$$\begin{aligned}\det(\mathbf{M}-\lambda\mathbf{I}) &= 0\\ (0.25-\lambda)(0.5-\lambda)-0.375 &= 0\\ 4\lambda^2-3\lambda-1 &= 0\\ (4\lambda+1)(\lambda-1) &= 0\\ \lambda &= -0.25 \text{ or } 1\end{aligned}$$

$\underline{\lambda_1 = -0.25:}$

$$\begin{pmatrix}0.25 & 0.5\\ 0.75 & 0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = -0.25\begin{pmatrix}x\\y\end{pmatrix}$$

$0.5x + 0.5y = 0 \Rightarrow x = -y$

$v_1 = \begin{pmatrix}-1\\1\end{pmatrix}$

$\underline{\lambda_2 = 1:}$

$\begin{pmatrix} 0.25 & 0.5 \\ 0.75 & 0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$

$-0.75x + 0.5y = 0 \Rightarrow 3x = 2y$

$v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

b So, $\mathbf{P} = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -0.25 & 0 \\ 0 & 1 \end{pmatrix}$

c

$$\begin{aligned}\mathbf{M}^n &= \mathbf{PD}^n\mathbf{P}^{-1} \\ &= \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -0.25 & 0 \\ 0 & 1 \end{pmatrix}^n\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} (-0.25)^n & 0 \\ 0 & 1^n \end{pmatrix}\frac{1}{5}\begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} -(-0.25)^n & 2 \\ (-0.25)^n & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3(-0.25)^n + 2 & -2(-0.25)^n + 2 \\ -3(-0.25)^n + 3 & 2(-0.25)^n + 3 \end{pmatrix}\end{aligned}$$

Since $(-0.25)^n$ approaches 0 as n becomes large,

$\mathbf{M}^n$ approaches $\frac{1}{5}\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{pmatrix}$

20 a

$$\begin{aligned}\det(\mathbf{M} - \lambda\mathbf{I}) &= 0 \\ (a-\lambda)^2 - b^2 &= 0 \\ \lambda &= a \pm b\end{aligned}$$

b $\underline{\lambda_1 = a + b:}$

$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (a+b)\begin{pmatrix} x \\ y \end{pmatrix}$

$-x + y = 0 \Rightarrow x = y$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = a - b:}$

$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (a-b)\begin{pmatrix} x \\ y \end{pmatrix}$

$x + y = 0 \Rightarrow y = -x$

$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$

d

$$\begin{aligned}\mathbf{M}^n &= \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a+b & 0\\ 0 & a-b\end{pmatrix}^n\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}^{-1}\\ &= \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}(a+b)^n & 0\\ 0 & (a-b)^n\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}(a+b)^n & (a+b)^n\\ (a-b)^n & -(a-b)^n\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}(a+b)^n+(a-b)^n & (a+b)^n-(a-b)^n\\ (a+b)^n-(a-b)^n & (a+b)^n+(a-b)^n\end{pmatrix}\end{aligned}$$

e $\mathbf{M}^n = \frac{1}{2}\begin{pmatrix}1+0.2^n & 1-0.2^n\\ 1-0.2^n & 1+0.2^n\end{pmatrix}$

Since 0.2^n approaches 0 as n becomes large,

$\mathbf{M}^n$ approaches $\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}0.5 & 0.5\\ 0.5 & 0.5\end{pmatrix}$

Mixed Practice 3

1 **a**

$$\begin{aligned}\mathbf{P} &= \mathbf{R}-\mathbf{F}\\ &= \begin{pmatrix}110 & 85\\ 154 & 102\\ 130 & 175\end{pmatrix} - \begin{pmatrix}42 & 30\\ 65 & 54\\ 88 & 106\end{pmatrix}\\ &= \begin{pmatrix}68 & 55\\ 89 & 48\\ 42 & 69\end{pmatrix}\end{aligned}$$

b $\mathbf{Q} = 1.05\mathbf{R} - 1.02\mathbf{F}$

c

$$\begin{aligned}\mathbf{Q} &= 1.05\begin{pmatrix}110 & 85\\ 154 & 102\\ 130 & 175\end{pmatrix} - 1.02\begin{pmatrix}42 & 30\\ 65 & 54\\ 88 & 106\end{pmatrix}\\ &= \begin{pmatrix}72.66 & 58.65\\ 95.40 & 52.02\\ 46.74 & 75.63\end{pmatrix}\end{aligned}$$

2

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix}2 & -3\\ 0 & 4\end{pmatrix}\begin{pmatrix}2k & -k\\ k & 1\end{pmatrix}\\ &= \begin{pmatrix}4k-3k & -2k-3\\ 4k & 4\end{pmatrix}\\ &= \begin{pmatrix}k & -2k-3\\ 4k & 4\end{pmatrix}\end{aligned}$$

3 **a** If A and B are commutative, then

$$\begin{pmatrix}2k & 0\\ k-1 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\ 3 & -1\end{pmatrix} = \begin{pmatrix}2 & 0\\ 3 & -1\end{pmatrix}\begin{pmatrix}2k & 0\\ k-1 & -1\end{pmatrix}$$
$$\begin{pmatrix}4k & 0\\ 2k-5 & 1\end{pmatrix} = \begin{pmatrix}4k & 0\\ 5k+1 & 1\end{pmatrix}$$

So

$$\begin{aligned} 2k - 5 &= 5k + 1 \\ 3k &= -6 \\ k &= -2 \end{aligned}$$

b For example, $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$\mathbf{CD} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{DC} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

So, $\mathbf{CD} \neq \mathbf{DC}$

4 $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$

$$= \frac{1}{-6 - (-5)}\begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$$

5

$$\begin{aligned} \mathbf{AC} &= \mathbf{B} \\ \mathbf{C} &= \mathbf{A}^{-1}\mathbf{B} \\ &= \frac{1}{-10 - (-6)}\begin{pmatrix} 2 & -3 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ k & 4 \end{pmatrix} \\ &= -\frac{1}{4}\begin{pmatrix} 2 - 3k & -6 \\ 2 - 5k & -14 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 3k - 2 & 6 \\ 5k - 2 & 14 \end{pmatrix} \end{aligned}$$

6 **a**

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{\det \mathbf{A}}\begin{pmatrix} 1 & -k \\ 0 & 3 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & -k \\ 0 & 3 \end{pmatrix} \end{aligned}$$

b

$$\begin{aligned} \mathbf{CA} &= \mathbf{B} \\ \mathbf{C} &= \mathbf{BA}^{-1} \\ &= \begin{pmatrix} 3 & k \\ 6 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & -k \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 2 & 1 - 2k \end{pmatrix} \end{aligned}$$

7 Let the original 2×4 matrix be $\mathbf{M}$.

Then

$$\mathbf{M}\begin{pmatrix} 9 & 18 & -13 & 3 \\ -3 & -6 & 4 & -1 \\ 7 & 12 & -15 & 2 \\ 1 & 1 & -13 & 1 \end{pmatrix} = \begin{pmatrix} 68 & 127 & -130 & 22 \\ 119 & 218 & -240 & 38 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 68 & 127 & -130 & 22 \\ 119 & 218 & -240 & 38 \end{pmatrix} \begin{pmatrix} 9 & 18 & -13 & 3 \\ -3 & -6 & 4 & -1 \\ 7 & 12 & -15 & 2 \\ 1 & 1 & -13 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 5 & 2 & 4 & 1 \\ 7 & 3 & 9 & 2 \end{pmatrix}$$

i.e. the two 4-digit ID numbers are 5241 and 7392

8 **a** $\mathbf{M}^2 = \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix}$

$$\mathbf{M}^{-1} = \frac{1}{-6—7} \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix}$$

So $\mathbf{M}^2 = \mathbf{M}^{-1}$

b

$$\begin{aligned} \mathbf{M}^2 &= \mathbf{M}^{-1} \\ \mathbf{M}^2\mathbf{M} &= \mathbf{M}^{-1}\mathbf{M} \\ \mathbf{M}^3 &= \mathbf{I} \end{aligned}$$

i.e. $\mathbf{M}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

c

$$\begin{aligned} \mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 7 \\ 23 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \mathbf{M}^{-1}\begin{pmatrix} 7 \\ 23 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 1 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 23 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -3 \end{pmatrix} \end{aligned}$$

i.e. $x = 2, y = -3$

9

$$\begin{aligned} \begin{pmatrix} 1.00 & 1.80 & 2.50 \\ 1.10 & 2.00 & 2.10 \\ 1.20 & 1.70 & 2.40 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} &= \begin{pmatrix} 24.9 \\ 22.9 \\ 24.3 \end{pmatrix} \\ \begin{pmatrix} A \\ B \\ C \end{pmatrix} &= \begin{pmatrix} 1.00 & 1.80 & 2.50 \\ 1.10 & 2.00 & 2.10 \\ 1.20 & 1.70 & 2.40 \end{pmatrix}^{-1}\begin{pmatrix} 24.9 \\ 22.9 \\ 24.3 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \end{aligned}$$

i.e. he orders 2 kg of A, 3 kg of B and 7 kg of C.

10 **a** **i**

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= 0 \\ (3-\lambda)(2-\lambda) - 20 &= 0 \\ \lambda^2 - 5\lambda - 14 &= 0 \end{aligned}$$

ii

$$\begin{aligned} (\lambda - 7)(\lambda + 2) &= 0 \\ \lambda &= 7 \text{ or } -2 \end{aligned}$$

b $\underline{\lambda_1 = 7:}$

$$\begin{pmatrix} 3 & -5 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$$

$-4x - 5y = 0 \Rightarrow 4x = -5y$

$v_1 = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

$\underline{\lambda_2 = -2:}$

$$\begin{pmatrix} 3 & -5 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

$5x - 5y = 0 \Rightarrow x = y$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

11

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= 0 \\ (3-\lambda)(4-\lambda) &= 0 \\ \lambda &= 3 \text{ or } 4 \end{aligned}$$

$\underline{\lambda_1 = 3:}$

$$\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$-2y = 0 \Rightarrow y = 0$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underline{\lambda_2 = 4:}$

$$\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$$

$-x - 2y = 0 \Rightarrow x = -2y$

$v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

12 $\mathbf{AB} + k\mathbf{C} = \mathbf{I}$

$$\begin{pmatrix} a & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & b \\ 0 & -4 \end{pmatrix} + k\begin{pmatrix} 1 & 4 \\ 2 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3a + k & ab + 4 + 4k \\ 6 + 2k & 2b - 4 + ck \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$\begin{cases} 3a + k = 1 & (1) \\ 6 + 2k = 0 & (2) \\ ab + 4 + 4k = 0 & (3) \\ 2b - 4 + ck = 1 & (4) \end{cases}$$

From (2): $k = -3$

Substituting into (1): $3a - 3 = 1 \Rightarrow a = \frac{4}{3}$

Substituting into (3): $\frac{4}{3}b + 4 - 12 = 0 \Rightarrow b = 6$

Substituting into (4): $12 - 4 - 3c = 1 \Rightarrow c = \frac{7}{3}$

13 a i

$$\begin{aligned} \det \mathbf{M} &= 0 \\ 24 - 5k &= 0 \\ k &= 4.8 \end{aligned}$$

ii

$$\begin{aligned} \mathbf{M}^{-1} &= \frac{1}{\det \mathbf{M}} \begin{pmatrix} 8 & -k \\ -5 & 3 \end{pmatrix} \\ &= \frac{1}{24 - 5k} \begin{pmatrix} 8 & -k \\ -5 & 3 \end{pmatrix} \end{aligned}$$

b

$$\begin{aligned} \begin{pmatrix} 3 & 7 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 \\ 13 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3 & 7 \\ 5 & 8 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 13 \end{pmatrix} \\ &= \frac{1}{24 - 35} \begin{pmatrix} 8 & -7 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 13 \end{pmatrix} \\ &= -\frac{1}{11} \begin{pmatrix} -99 \\ 44 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ -4 \end{pmatrix} \end{aligned}$$

i.e. $x = 9, y = -4$

14 a

$$\begin{aligned} \mathbf{ABC} &= \mathbf{I} \\ \mathbf{A}^{-1}\mathbf{ABC} &= \mathbf{A}^{-1}\mathbf{I} && \text{(multiplying on the left by } \mathbf{A}^{-1}\text{)} \\ \mathbf{BC} &= \mathbf{A}^{-1} \\ \mathbf{B}^{-1}\mathbf{BC} &= \mathbf{B}^{-1}\mathbf{A}^{-1} && \text{(multiplying on the left by } \mathbf{B}^{-1}\text{)} \\ \mathbf{C} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ \mathbf{CA} &= \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{A} && \text{(multiplying on the right by } \mathbf{A}\text{)} \\ \mathbf{CA} &= \mathbf{B}^{-1} \end{aligned}$$

b

$$\begin{aligned} \mathbf{B}^{-1} &= \mathbf{CA} \\ &= \begin{pmatrix} 0 & -1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ -2 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ -20 & -27 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{B} &= (\mathbf{B}^{-1})^{-1} \\ &= \frac{1}{6} \begin{pmatrix} -27 & -3 \\ 20 & 2 \end{pmatrix} \end{aligned}$$

15

$$\mathbf{M} + \mathbf{M}^{-1} = \mathbf{I}$$

$$\begin{pmatrix} x & 2 \\ y & 0 \end{pmatrix} + \frac{1}{-2y}\begin{pmatrix} 0 & -2 \\ -y & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x & 2 + \frac{1}{y} \\ y + \frac{1}{2} & -\frac{x}{2y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$\begin{cases} x = 1 & (1) \\ 2 + \frac{1}{y} = 0 & (2) \\ y + \frac{1}{2} = 0 & (3) \\ -\frac{x}{2y} = 1 & (4) \end{cases}$$

From (3): $y = -\frac{1}{2}$

Check $x = 1, y = -\frac{1}{2}$ in (2) and (4):

$$2 + \frac{1}{-\frac{1}{2}} = 0$$

$$-\frac{1}{2\left(-\frac{1}{2}\right)} = -\frac{1}{-1} = 1$$

Both consistent, so $x = 1, y = -\frac{1}{2}$

16

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} 4 & -3 \\ 2 & 9 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} 15 & -2 \\ -3 & 20 \end{pmatrix} \end{aligned}$$

17 a

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= 0 \\ (3-\lambda)(9-\lambda) - (-8) &= 0 \\ \lambda^2 - 12\lambda + 35 &= 0 \\ (\lambda - 5)(\lambda - 7) &= 0 \\ \lambda &= 5 \text{ or } 7 \end{aligned}$$

$\underline{\lambda_1 = 5:}$

$$\begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$-2x + 4y = 0 \Rightarrow x = 2y$$

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = 7:}$

$$\begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix}$$

$$-4x + 4y = 0 \Rightarrow x = y$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So, $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$

b

$$\begin{aligned}
\mathbf{M}^n &= \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}^n\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & 7^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2(5^n) & 7^n \\ 5^n & 7^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2(5^n) - 7^n & 2(7^n) - 2(5^n) \\ 5^n - 7^n & 2(7^n) - 5^n \end{pmatrix}
\end{aligned}$$

18 a i

$$\begin{aligned}
\det(\mathbf{M} - \lambda\mathbf{I}) &= 0 \\
(a-\lambda)(b-\lambda) - (1-a)(1-b) &= 0 \\
\lambda^2 - (a+b)\lambda - (1-a-b) &= 0 \\
(\lambda - 1)\big(\lambda + (1-a-b)\big) &= 0 \\
\lambda &= 1 \text{ or } a+b-1
\end{aligned}$$

ii

$\underline{\lambda_1 = 1:}$

$$\begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$(a-1)x + (1-b)y = 0 \Rightarrow (a-1)x = (b-1)y$$

$$v_1 = \begin{pmatrix} b-1 \\ a-1 \end{pmatrix}$$

$\lambda_2 = a + b - 1:$

$$\begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (a+b-1)\begin{pmatrix} x \\ y \end{pmatrix}$$

$$(1-b)x + (1-b)y = 0 \Rightarrow x = -y$$

$$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b So, $\mathbf{P} = \begin{pmatrix} b-1 & 1 \\ a-1 & -1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & a+b-1 \end{pmatrix}$

c

$$
\begin{aligned}
\mathbf{M}^n &= \mathbf{PD}^n\mathbf{P}^{-1}\\
&= \begin{pmatrix} b-1 & 1\\ a-1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & a+b-1\end{pmatrix}^n\begin{pmatrix} b-1 & 1\\ a-1 & -1\end{pmatrix}^{-1}\\
&= \begin{pmatrix} b-1 & 1\\ a-1 & -1\end{pmatrix}\begin{pmatrix}1^n & 0\\ 0 & (a+b-1)^n\end{pmatrix}\frac{1}{1-b-a+1}\begin{pmatrix} -1 & -1\\ 1-a & b-1\end{pmatrix}\\
&= \frac{1}{2-a-b}\begin{pmatrix} b-1 & (a+b-1)^n\\ a-1 & -(a+b-1)^n\end{pmatrix}\begin{pmatrix} -1 & -1\\ 1-a & b-1\end{pmatrix}\\
&= \frac{1}{2-a-b}\begin{pmatrix} 1-b+(1-a)(a+b-1)^n & 1-b-(1-b)(a+b-1)^n\\ 1-a-(1-a)(a+b-1)^n & 1-a+(1-b)(a+b-1)^n\end{pmatrix}
\end{aligned}
$$

Note: $0 < a < 1$ and $0 < b < 1 \Rightarrow -1 < a+b-1 < 1$

So, as n becomes large $(a+b-1)^n$ approaches 0 and therefore $\mathbf{M}^n$ approaches

$\frac{1}{2-a-b}\begin{pmatrix}1-b & 1-b\\ 1-a & 1-a\end{pmatrix}$.

Worked solutions

4 Geometry and trigonometry

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 4A

25 Perimeter $= r\theta + 2r = 20.8$ cm

$$\text{Area} = \frac{1}{2}r^2\theta = 19.2 \text{ cm}^2$$

26 Perimeter $= r\theta + 2r = 27.9$ cm

$$\text{Area} = \frac{1}{2}r^2\theta = 48.1 \text{ cm}^2$$

27 Arc length $= r\theta = 1.2r = 12.3$

$$r = \frac{12.3}{1.2} = 10.3 \text{ cm}$$

28 **a** Arc length $= r\theta = 5\theta = 7$

$$\theta = \frac{7}{5} = 1.4 \text{ radians}$$

b Area $= \frac{1}{2}r^2\theta = 17.5$ cm^2

29 Area $= \frac{1}{2}r^2\theta = \frac{1}{2}(23)^2\theta = 185$

$$\theta = \frac{2 \times 185}{23^2} = 0.699 \text{ radians}$$

30 Area $= \frac{1}{2}r^2\theta = \frac{1}{2}(2.7)r^2 = 326$

$$r = \sqrt{\frac{2 \times 326}{2.7}} = 15.5 \text{ cm}$$

31 **a** Area $= \frac{1}{2}r^2\theta = \frac{1}{2}(1.3)r^2 = 87.3$

$$r = \sqrt{\frac{2 \times 87.3}{1.3}} = 11.6 \text{ cm}$$

b Perimeter $= r\theta + 2r = 38.2$ cm

32 Sector Area $= \frac{1}{2}(0.9)(7^2) = 22.05$ cm^2

Rectangle Area $= 7 \times 5 = 35$ cm^2

Total Area $= 57.1$ cm^2

Arc length $= 0.9 \times 7 = 6.3$ cm

Straight perimeter $= 2 \times 5 + 2 \times 7 = 24$ cm

Total perimeter $= 30.3$ cm

33 Perimeter $= r\theta + 2r = 3.8r = 26$ cm

$$r = \frac{26}{3.8} = 6.84 \text{ cm}$$

34 Perimeter $= r\theta + 2r = (2 + \theta)r = 30$ cm (1)

Area $= \frac{1}{2}r^2\theta = 18 \text{ cm}^2$ (2)

$$(1): \theta = \frac{30}{r} - 2$$

Substituting into (2):

$$r^2\left(\frac{30}{r} - 2\right) = 36$$

$$2r^2 - 30r + 36 = 0$$

$$r^2 - 15r + 18 = 0$$

$$r = \frac{15 \pm \sqrt{(-15)^2 - 4(1)(18)}}{2(1)}$$
$$= 1.32 \text{ cm or } 13.7 \text{ cm}$$

Corresponding values of θ would be $\theta = \frac{30}{r} - 2 = 20.8$ or 0.192

In context, $\theta \leq 2\pi$ so reject the first solution.

$r = 13.7$ cm is the only valid solution.

35 Each of the sides is a 60° ($\frac{\pi}{3}$ radian) arc centred at the opposite vertex.

$$\text{Perimeter} = 3 \times \frac{\pi}{3} \times 12 = 12\pi \text{ cm}$$

36 Arc length $= r\theta = 8 \times 0.9 = 7.2$ cm

Cosine rule:

$$PQ = \sqrt{8^2 + 8^2 - 2(8)(8)\cos 0.9} = 6.96 \text{ cm}$$

Difference in lengths is $7.2 - 6.96 = 0.241$ cm

37 Arc length $= r\theta = 12 \times 0.6 = 7.2$ cm

Cosine rule:

$$AB = \sqrt{12^2 + 12^2 - 2(12)(12)\cos 0.6} = 7.09 \text{ cm}$$

Perimeter of shaded region $= 7.2 + 7.09 = 14.3$ cm

$$\text{Sector area} = \frac{1}{2}r^2\theta = \frac{1}{2}(12)^2(0.6) = 43.2 \text{ cm}^2$$

Sine rule for area:

Triangle OAB area $= \frac{1}{2}(12)(12)\sin 0.6 = 40.7\ \text{cm}^2$

Shaded region $= 43.2 - 40.7 = 2.55\ \text{cm}^2$

38 **a** Sector area $= \frac{1}{2}r^2\theta = 8\theta$

Sine rule for area:

Triangle OPQ area $= \frac{1}{2}(4)(4)\sin\theta = 8\sin\theta$

Shaded area $= 8(\theta - \sin\theta) = 6$

$\theta - \sin\theta = \frac{6}{8} = 0.75$

b From the GDC, $\theta = 1.74$ radians

c Perimeter of shaded region $= r\theta + \sqrt{2r^2 - 2r^2\cos\theta} = 13.1\ \text{cm}$

39 The shaded region is the difference between two sector areas:

Shaded area $= \frac{1.2}{2}((15+x)^2 - 15^2)$

$0.6(x^2 + 30x) = 59.4$

$x^2 + 30x - 99 = 0$

$(x-3)(x+33) = 0$

$x = 3$ is the only positive solution

40 Radius r of the sector becomes the slant length of the cone

$r = \sqrt{22^2 + 8^2} = 23.4\ \text{cm}$

Arclength of the sector becomes the circumference of the cone base

$r\theta = 2\pi \times 8$

$\theta = \frac{16\pi}{r} = 2.15$ radians

41 If the two circle intersections are P and Q then PC_1C_2 is equilateral, since each side is a radius of one circle.

Then $P\widehat{C_1}Q = \frac{2\pi}{3}$ radians

The shaded region is the sum of two identical segments.

Segment area $= \frac{r^2}{2}(\theta - \sin\theta)$

Shaded area $= r^2(\theta - \sin\theta) = 8^2\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right) = 8^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) = 78.6\ \text{cm}^2$

Perimeter $= 2r\theta = \frac{32\pi}{3} = 33.5\ \text{cm}$

Exercise 4B

17 $\cos\theta = \pm\sqrt{1-\sin^2\theta} = \pm\sqrt{1-\frac{16}{81}} = \pm\frac{\sqrt{65}}{9}$

18 $\frac{\pi}{2} < x < \frac{3\pi}{2}$ so $\sin x < 0$ and $\tan x > 0$

a $\sin x = -\sqrt{1-\cos^2 x} = -\frac{\sqrt{21}}{5}$

b $\tan x = \frac{\sin x}{\cos x} = \frac{\sqrt{21}}{2}$

19 $\frac{\pi}{2} < x < \pi$ so $\cos x < 0$ and $\tan x < 0$

a $\cos x = -\sqrt{1-\sin^2 x} = -\frac{\sqrt{40}}{7} = -\frac{2\sqrt{10}}{7}$

b $\tan x = \frac{\sin x}{\cos x} = -\frac{3}{\sqrt{40}} = -\frac{3\sqrt{10}}{20}$

20 a Maximum d is $5 + 1.6 = 6.6$ m

b Require $\sin\left(\frac{\pi t}{12}\right) = 1$

$$\frac{\pi t}{12} = \frac{\pi}{2}$$

$t = 6$

First high tide is at 6 a.m.

21 a $\frac{\pi t}{4} = 2\pi \Rightarrow t = 8$

One revolution takes 8 minutes.

b $6.2 + 4.8 = 11$ m

c $h\left(\frac{8}{3}\right) = 6.2 - 4.8\cos\left(\frac{2\pi}{3}\right) = 8.6$ m

22 a Maximum h is $1.4 + 0.2 = 1.6$ m

b From the GDC, there are 7 complete oscillations in 3 seconds.

c From the GDC, the ball is 1.5m above ground for the second time at 0.279 seconds

23 Sine rule for area

$$35 = \frac{1}{2}(8)(11)\sin\theta°$$

$$\sin\theta° = \frac{35}{44}$$

$\theta_1 = 52.7, \theta_2 = 180 - \theta_1 = 127$

24 Cosine rule:

$5.3^2 = 7.5^2 + AC^2 - 2(7.5)(AC)\cos 44°$

$AC^2 - 2(7.5)(AC)\cos 44° + 7.5^2 - 5.3^2 = 0$

$AC = 7.5\cos 44° \pm \sqrt{(7.5\cos 44°)^2 + (5.3^2 - 7.5^2)}$
$= 4.42$ cm or 6.37 cm

25 Cosine rule:

$$(KM)^2 = (KL)^2 + (LM)^2 - 2(KL)(LM)\cos(KLM)$$
$$225 = 12^2 + (LM)^2 - 2(12)(LM)\cos 55°$$
$$(LM)^2 - 13.8(LM) - 81 = 0$$

This quadratic has one positive and one negative solution; in context, only the positive solution makes sense so LM has only one possible length.

$$LM = \frac{13.8 + \sqrt{13.8^2 + 324}}{2} = 18.2 \text{ cm}$$

26 $3\sin^2 x + 7\cos^2 x = 3\sin^2 x + 7(1 - \sin^2 x) = 7 - 4\sin^2 x$

27 $4\cos^2 x - 5\sin^2 x = 4\cos^2 x - 5(1 - \cos^2 x) = 9\cos^2 x - 5$

28 $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

$$\equiv (\sin^2 x + 2\sin x\cos x + \cos^2 x) + (\sin^2 x - 2\sin x\cos x + \cos^2 x)$$
$$\equiv 2(\sin^2 x + \cos^2 x)$$
$$\equiv 2$$

29

$$1 + \tan^2\theta \equiv 1 + \frac{\sin^2\theta}{\cos^2\theta}$$
$$\equiv \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$
$$\equiv \frac{1}{\cos^2\theta}$$

30

$$\frac{1}{\sin^2 A} - \frac{1}{\tan^2 A} \equiv \frac{1}{\sin^2 A} - \frac{1}{\tan^2 A}$$
$$\equiv \frac{1}{\sin^2 A} - \frac{\cos^2 A}{\sin^2 A}$$
$$\equiv \frac{1 - \cos^2 A}{\sin^2 A}$$
$$\equiv \frac{\sin^2 A}{\sin^2 A}$$
$$\equiv 1$$

31

$$\frac{1}{\cos\theta} - \cos\theta \equiv \frac{1 - \cos^2\theta}{\cos\theta}$$
$$\equiv \frac{\sin^2\theta}{\cos\theta}$$
$$\equiv \sin\theta\frac{\sin\theta}{\cos\theta}$$
$$\equiv \sin\theta\tan\theta$$

32

$$2\cos^2 x + \sin x = k$$

$$2(1 - \sin^2 x) + \sin x - k = 0$$

$$2\sin^2 x - \sin x + k - 2 = 0$$

Quadratic in $\sin x$; using quadratic formula,

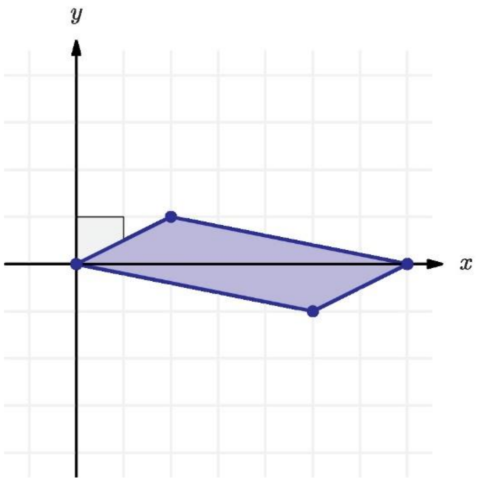
$$\sin x = \frac{1 \pm \sqrt{1^2 - 4(2)(k-2)}}{4} = \frac{1 \pm \sqrt{17 - 8k}}{4}$$

Exercise 4C

35 **a** $\begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$

The image is $(8, -3)$

b



c $\det \mathbf{M} = -7$ so the area of the image of the unit square is 7

36 **a** $\begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ p \end{pmatrix} = \begin{pmatrix} q \\ -1 \end{pmatrix}$

$$\begin{cases} -3 + p = q & (1) \\ 9 - 2p = -1 & (2) \end{cases}$$

$(2): p = 5$

$(1): q = 2$

b Original triangle has area $\frac{1}{2} \times 6 \times 5 = 15$

$\det \mathbf{A} = -1$

The image also has area 15

c From the GDC, $\mathbf{A}^3 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -9 \end{pmatrix}$

The image point has coordinates $(4\,, -9)$

37 **a** $p = 2, q = 3$

b $\mathbf{R}$ has matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} b-3 \\ 1-a \end{pmatrix}$

The point is mapped to $(b - 3, 1 - a)$

c $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\left(\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}-3\\1\end{pmatrix}\right)=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}a-3\\b+1\end{pmatrix}=\begin{pmatrix}b+1\\3-a\end{pmatrix}$

The point is now mapped to $(b+1, 3-a)$, which cannot be the same point for any point (a, b)

38 a Anticlockwise rotation 135°: $\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & -1\\1 & -1\end{pmatrix}$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{5}{\sqrt{2}}\end{pmatrix}$$

The image is at $\left(\frac{\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right)$

b A subsequent enlargement with scale factor 4 will produce image point $(2\sqrt{2}, 10\sqrt{2})$

c The determinant of a rotation is 1 a the determinant of an enlargement with scale factor 4 is $4^2=16$, so the area of the image square is 16

39 a A reflection in the line $y=(\tan\theta)x$ has matrix $\mathbf{M}=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\\sin 2\theta & -\cos 2\theta\end{pmatrix}$

For $\theta=\arctan 3\approx 1.25$, $\mathbf{M}=\begin{pmatrix}-0.8 & 0.6\\0.6 & 0.8\end{pmatrix}$

Comment: You can find this directly from the GDC or by using identities:

$\tan\theta=3$, so $\sec^2\theta=1+\tan^2\theta=10$

Then

$$\cos 2\theta=2\cos^2\theta-1=\frac{2}{\sec^2\theta}-1=-\frac{4}{5}$$

$$\sin 2\theta=2\sin\theta\cos\theta=\frac{2\tan\theta}{\sec^2\theta}=\frac{3}{5}$$

b

$$\begin{pmatrix}-0.8 & 0.6\\0.6 & 0.8\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix}=\begin{pmatrix}-0.4\\2.8\end{pmatrix}$$

The image point is $(-0.4, 2.8)$

40 a $\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}7\\20\end{pmatrix}$

The image point is $(7, 20)$

b $\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-2\\2\end{pmatrix}$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}^{-1}\begin{pmatrix}-2\\2\end{pmatrix}=\begin{pmatrix}-2.4\\2.8\end{pmatrix}$$

41 a Matrices for the transformations are

$$\mathbf{N}=\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\ \mathbf{R}=\begin{pmatrix}\cos 30° & -\sin 30°\\\sin 30° & \cos 30°\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\\1 & \sqrt{3}\end{pmatrix}$$

$$\mathbf{NR}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\sqrt{3}-1\\2+\sqrt{3}\end{pmatrix}$$

The image has coordinates $\left(2\sqrt{3}-1, 2+\sqrt{3}\right)$

b $\mathbf{RN} = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$

c

$\mathbf{NR} = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} = \mathbf{RN}$, which shows that the order of these two transformations does not affect the result.

42 Anticlockwise rotation 30°: $\mathbf{R} = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$

Anticlockwise rotation 60°: $\mathbf{S} = \begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

$\mathbf{RS} = \frac{1}{4}\begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix}$

43 **a** Reflection in x-axis: $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Anti-clockwise rotation 90°: $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

S followed by **R**: $\mathbf{RS} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

A reflection in the line $y = (\tan\theta)x$ has matrix $\mathbf{M} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

When $\theta = \frac{\pi}{4}$, $\cos 2\theta = 0$ and $\sin 2\theta = 1$

$\tan\left(\frac{\pi}{4}\right) = 1$

RS represents a reflection through $y = x$

b **R** followed by **S**: $\mathbf{SR} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

A reflection in the line $y = (\tan\theta)x$ has matrix $\mathbf{M} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

When $\theta = \frac{3\pi}{4}$, $\cos 2\theta = 0$ and $\sin 2\theta = -1$

$\tan\left(\frac{3\pi}{4}\right) = -1$

RS represents a reflection through $y = -x$

44 **a** Anticlockwise rotation 120°: $\mathbf{R} = \begin{pmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

Enlargement scale factor $\frac{1}{2}$: $\mathbf{S} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbf{T} = \mathbf{SR} = \frac{1}{4}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

$\mathbf{T}^2 = \frac{1}{16}\begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$

b

D is the image of B under $\mathbf{T}$: $\frac{1}{4}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}0\\8\end{pmatrix} = \begin{pmatrix}-2\sqrt{3}\\-2\end{pmatrix}$ so D has coordinates $(-2\sqrt{3}, -2)$

F is the image of B under $\mathbf{T}^2$: $\frac{1}{8}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}\begin{pmatrix}0\\8\end{pmatrix} = \begin{pmatrix}\sqrt{3}\\-1\end{pmatrix}$ so D has coordinates $(\sqrt{3}, -1)$

$\det \mathbf{T} = \frac{1}{4}$

Triangle AOB has area 32 so OCD has area 8 and OEF has area 2

The total area is therefore 42.

45 a $\mathbf{M}^2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$ so A has coordinates $(4, 4)$

b $\det \mathbf{M} = 2$ so $\det \mathbf{M}^6 = 2^6 = 64$

The image of the unit square under $\mathbf{M}^6$ has area 64

46 a A reflection in the line $y = (\tan\theta)x$ has matrix $\mathbf{S} = \begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$

$\tan\theta = -1$ so $\theta = -\frac{\pi}{4}$ then $\mathbf{S} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$

b $\det(\mathbf{S} - \lambda\mathbf{I}) = 0$

$\det\begin{pmatrix}-\lambda & -1\\ -1 & -\lambda\end{pmatrix} = 0$

$\lambda^2 = 1$

$\lambda = \pm 1$

Eigenvalue $\lambda_1 = 1$:

$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$

$\begin{cases}-x = y\\ -y = x\end{cases}$

Eigenvector $\mathbf{v_1} = \begin{pmatrix}-1\\1\end{pmatrix}$

Eigenvalue $\lambda_2 = -1$:

$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = -\begin{pmatrix}x\\y\end{pmatrix}$

$\begin{cases}-x = -y\\ -y = -x\end{cases}$

Eigenvector $\mathbf{v_2} = \begin{pmatrix}1\\1\end{pmatrix}$

c As a reflection, the eigenvalues are -1 and 1, with eigenvectors perpendicular to and parallel to the line of reflection, respectively. Points on the line $y = -x$ are unaffected by the transformation; points in the line $y = x$ are reflected through the origin.

47 a $\det(\mathbf{B_1} - \lambda \mathbf{I}) = 0$

$\det\begin{pmatrix} -4-\lambda & 1 \\ 3 & -2-\lambda \end{pmatrix} = 0$

$\lambda^2 + 6\lambda + 5 = 0$

$(\lambda + 1)(\lambda + 5) = 0$

Eigenvalue $\lambda_1 = -1$:

$\begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{cases} -4x + y = -x \Rightarrow y = 3x \\ 3x - 2y = -y \Rightarrow y = 3x \end{cases}$

Eigenvector $\mathbf{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Eigenvalue $\lambda_2 = -5$:

$\begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{cases} -4x + y = -5x \Rightarrow y = -x \\ 3x - 2y = -5y \Rightarrow y = -x \end{cases}$

Eigenvector $\mathbf{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

b The invariant lines are $y = 3x$ and $y = -x$

c Seeking eigenvalues for $\mathbf{B_2}$:

$\det(\mathbf{B_2} - \lambda \mathbf{I}) = 0$

$\det\begin{pmatrix} -5-\lambda & -3 \\ 4 & 1-\lambda \end{pmatrix} = 0$

$\lambda^2 + 4\lambda + 7 = 0$

$\lambda = 2 \pm \sqrt{-3}$

Since there are no real eigenvalues, there are no real eigenvectors and hence no invariant lines through the origin.

48 a Line has equation $y = 2x = x\tan\theta$

$\tan\theta = 2$

$\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}} = \dfrac{1}{\sqrt{5}}$

$\sin\theta = \sqrt{1-\cos^2\theta} = \dfrac{2}{\sqrt{5}}$

The transformation is given by $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = -\dfrac{3}{5}$

$\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{4}{5}$

$\mathbf{R} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$

$\mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\mathbf{T} = \mathbf{SR} = \frac{1}{5}\begin{pmatrix} -4 & -3 \\ -3 & 4 \end{pmatrix}$

b Let invariant line under **T** be $y = ax$

Then a point on an invariant line is $\begin{pmatrix} k \\ ka \end{pmatrix}$

$\mathbf{T}\begin{pmatrix} k \\ ka \end{pmatrix} = \frac{k}{5}\begin{pmatrix} -4 - 3a \\ -3 + 4a \end{pmatrix}$

Require that this new position is also on the line; $-3 + 4a = a(-4 - 3a)$

$3a^2 + 8a - 3 = 0$

$(3a - 1)(a + 3) = 0$

$a = \frac{1}{3}$ or -3

The two invariant lines are $y = \frac{1}{3}x$ and $y = -3x$

c **T** has general form $\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$ so must be a reflection.

where $\cos 2\phi = -\frac{4}{5}, \sin 2\phi = -\frac{3}{5}$

The line of reflection $y = \tan\phi$ must be one of the two invariant lines.

Since $\sin 2\phi < 0, \pi < 2\phi < \frac{3\pi}{2}$ so $\frac{\pi}{2} < \phi < \frac{3\pi}{4}$

Then $\tan\phi < -1$

The line of reflection must be $y = -3x$

49 a P_1 has coordinates $\left(\frac{1}{2}x + 1, \frac{1}{2}y\right)$

b P_2 has coordinates $\left(\frac{1}{4}x + \frac{3}{2}, \frac{1}{4}y\right)$

P_3 has coordinates $\left(\frac{1}{8}x + \frac{7}{4}, \frac{1}{8}y\right)$

P_4 has coordinates $\left(\frac{1}{16}x + \frac{15}{8}, \frac{1}{16}y\right)$

c P_n has coordinates $\left(\frac{1}{2^n}x + \frac{2^n - 1}{2^{n-1}}, \frac{1}{2^n}y\right)$

d The original triangle T_0 has area $\frac{1}{2}$

Then T_n has area $\frac{1}{2} \times \frac{1}{4^n}$, since the enlargement causes the area to reduce to 25% each time.

Total area $= \sum_{n=0}^{\infty} \frac{1}{2} \times \frac{1}{4^n}$

This is the sum of a geometric series, with $a = 0.5$ and $r = 0.25$

$$S_\infty = \frac{a}{1-r} = \frac{0.5}{0.75} = \frac{2}{3}$$

50 a In step 1, the side length is $\frac{1}{2}$ and the side length halves each time, so in step 4 the side length is $\frac{1}{16}$

b The area at the start is $A_0 = 1$

At step 1, 3 new squares are added, each with side length $\frac{1}{2}$ and so area $\frac{1}{4}$

At each step $n > 1$, 9 new squares are added, each with side length $\frac{1}{2^n}$ and so area $\frac{1}{4^n}$

If A_n is the new area added at step $n > 1$ then $A_n = 9 \times \frac{1}{4^n}$

$$\text{Total area} = 1 + \frac{3}{4} + 9\sum_{n=2}^{\infty}\left(\frac{1}{4}\right)^n = \frac{7}{4} + \frac{9}{16}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

The sum if for a geometric series, with $a = \frac{9}{16}$ and $r = \frac{1}{4}$

$$\text{Total area} = \frac{7}{4} + \frac{a}{1-r} = \frac{7}{4} + \frac{\left(\frac{9}{16}\right)}{\left(\frac{3}{4}\right)} = \frac{7}{4} + \frac{3}{4} = \frac{5}{2}$$

Mixed Practice

1 a Amplitude is 1.3 m

b Period $= \frac{2\pi}{2.5} = 2.51$ m

2 Sine rule:

$$\sin\theta = \frac{6\sin 38°}{4.5}$$

$\theta = 55.2°$ or $125°$

3 a Obtuse angle: $90° < A < 180°$

$\cos A < 0$ for angle in this interval.

$$\cos A = -\sqrt{1-\sin^2 A} = -\sqrt{1-\frac{25}{169}} = -\frac{12}{13}$$

b $\tan A \equiv \frac{\sin A}{\cos A} = -\frac{5}{12}$

4 a $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

The image point has coordinates $(4, 7)$

b $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

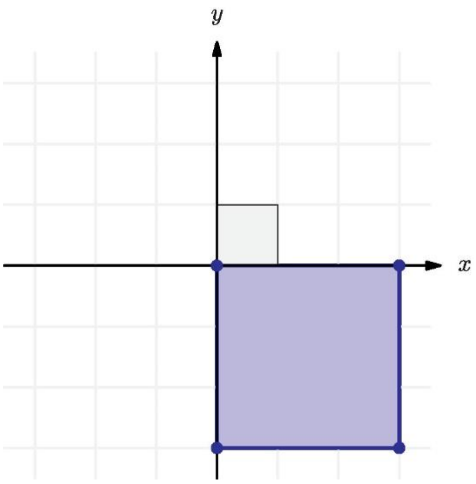
The source point has coordinates $(-1, 4)$

c $\det \mathbf{M} = -1$ so the area is unchanged, and remains 1.

5 **a** Anticlockwise rotation $-90°$: $\mathbf{R} = \begin{pmatrix} \cos(-90°) & -\sin(-90°) \\ \sin(-90°) & \cos(-90°) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

b $\mathbf{T} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{R} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$

c



6 **a** **R** has matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

S has matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbf{SR} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$

b $\mathbf{RS} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, not the same transformation.

7 $\sin\left(\frac{x}{2}\right)$ has period 4π.

$\cos\left(\frac{x}{5}\right)$ has period 10π.

The lowest common multiple of the two (where periods values are equal after integer multiplication) is 20π, so the period of any composite function of the two will be 20π.

8 $\mathrm{f}(x) = (4\cos(x - \pi) - 1)^2$

Let $\mathrm{g}(x) = 4\cos(x - \pi) - 1$ so that $\mathrm{f}(x) = \left(\mathrm{g}(x)\right)^2$

$\mathrm{g}(x)$ has range $-5 \le g(x) \le 3$

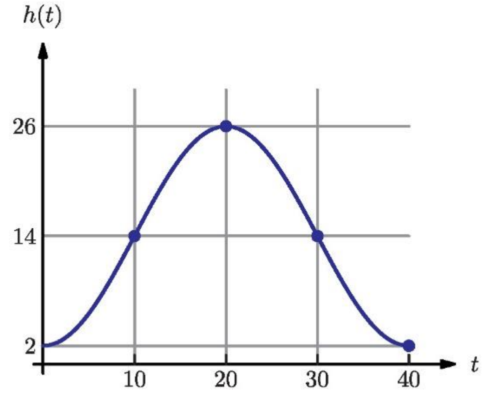
So $\mathrm{f}(x)$ has range $0 \le f(x) \le 25$

9 **a** **i** 14 m

ii 26 m

b Complete rotation takes 40 seconds, so the seat is at A after 10 seconds and at B after 30 seconds.

c

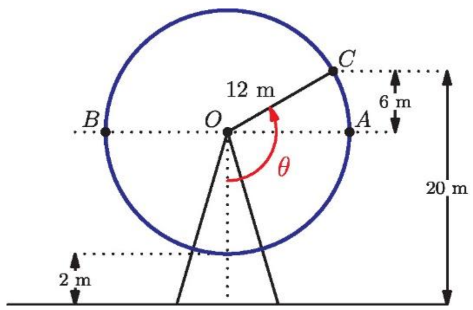


d **i** Amplitude $a = 12$

ii Frequency $b = \frac{2\pi}{\text{period}} = \frac{2\pi}{40} = \frac{\pi}{20}$

iii Central value $c = 14$

e **i**



ii Angle $\theta = 90° + \sin^{-1}\left(\frac{6}{12}\right) = 120°$

iii $bT = \frac{2\pi}{3}$

$T = \frac{2\pi}{3b} = \frac{40}{3}$

$T = 13.3$ seconds

10 **a** A reflection in the line $y = (\tan\theta)x$ has matrix $\mathbf{R} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

If $\theta = 0°$ then $\mathbf{R_0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

If $\theta = 30°$ then $\mathbf{R_{30}} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

$\mathbf{M} = \mathbf{R_{30}R_0} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$

b An anticlockwise rotation of θ about the origin has matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\mathbf{M} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix}$

M is an anticlockwise rotation of 60° about the origin.

c $\mathbf{R_0 R_{30}} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{pmatrix}$

The transformation obtained by reversing the order of the reflections is a clockwise rotation of $60°$ about the origin.

11 a $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

b $\mathbf{C} = \mathbf{BA} = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$

c $\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 6 \\ 12 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 12 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$$

The object coordinates are $(-4, 6)$.

12 Sine rule:

$$\sin A\hat{C}B = \frac{10\sin 40°}{7}$$

$A\hat{C}B = 66.7 \text{ or } 113°$

$B\hat{A}C = 180 - 40 - A\hat{C}B = 73.3° \text{ or } 26.7°$

Sine rule for area:

$$\begin{aligned}\text{Area } ABC &= \frac{1}{2}(AB)(AC)\sin B\hat{A}C \\ &= 33.5 \text{ cm}^2 \text{ or } 15.7 \text{ cm}^2\end{aligned}$$

The difference in area of the two possible triangles is $33.5 - 15.7 = 17.8 \text{ cm}^2$

13 If the two endpoints of the circle arc are PQ and it is assumed that the centre O of the circle from which the sector is taken lies at the midpoint of the right side of the rectangle then the height of the rectangle is the length of chord PQ and the width of the rectangle equals the radius of the circle.

$$\text{Sector area} = \frac{1}{2}r^2\theta = \frac{1}{2}r^2 = 7$$

$r = \sqrt{14} = 3.74 \text{ cm} \approx 37 \text{ mm}$

Cosine rule in triangle OPQ

$$\begin{aligned}PQ &= \sqrt{OP^2 + OQ^2 - 2(OP)(OQ)\cos P\hat{O}Q} \\ &= \sqrt{r^2 + r^2 - 2r^2\cos 1} \\ &= 3.59 \text{ cm} \approx 36 \text{ mm}\end{aligned}$$

Challenge to student:

Find the height and width of the rectangle with least area which can contain this sector.

14

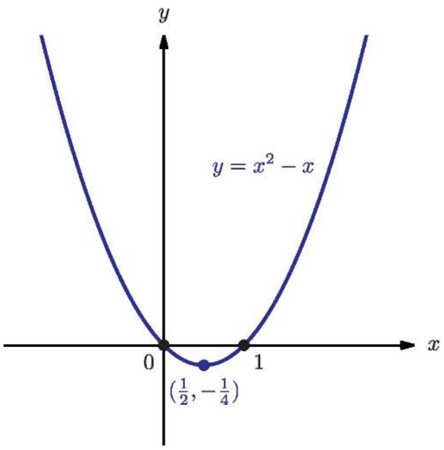
$$\frac{1}{1+\cos x}+\frac{1}{1-\cos x}\equiv\frac{(1-\cos x)+(1+\cos x)}{(1+\cos x)(1-\cos x)}$$
$$\equiv\frac{2}{1-\cos^2 x}$$
$$\equiv\frac{2}{\sin^2 x}$$

15 In the double period, there will be four solutions.

If $k=\arcsin x$ then the other three solutions are $\pi-k, 2\pi+k, 3\pi-k$

The sum of these four solutions is $k+(\pi-k)+(2\pi+k)+(3\pi-k)=6\pi$

16 a Positive quadratic, roots at 0 and 1, vertex at $x=\frac{1}{2}$.



b If $\sin^2 x-\sin x=k$ then there are only solutions for $-1\le\sin x\le 1$

The range of the function in part **a** for domain restricted to $[-1,1]$ is $[-0.25,2]$ as shown in the diagram.

So there for solutions to the equation $\sin^2 x-\sin x=k$, k must lie in the interval $-0.25\le k\le 2$

17 a An anticlockwise rotation of θ about the origin has matrix $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

$$\mathbf{R}=\begin{pmatrix}\cos 30° & -\sin 30°\\ \sin 30° & \cos 30°\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{pmatrix}$$

A reflection in the line $y=(\tan\theta)x$ has matrix $\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$

If $\theta=90°$ then $\mathbf{S}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

$$\mathbf{T}=\mathbf{R}^{-1}\mathbf{SR}=\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$$

b $\mathbf{T}=\mathbf{R}^{-1}\mathbf{SR}=\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}=\begin{pmatrix}\cos 120° & \sin 120°\\ \sin 120° & -\cos 120°\end{pmatrix}$

This is a reflection in the line $y=(\tan 60°)x$, $y=x\sqrt{3}$

18 A reflection in the line $y = (\tan\theta)x$ has matrix $\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$

If $\theta = 45°$, then $\mathbf{M} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

An anticlockwise rotation of θ about the origin has matrix $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

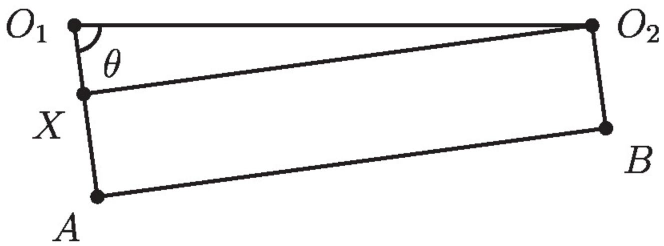
$$\mathbf{N} = \begin{pmatrix}\cos 45° & -\sin 45°\\ \sin 45° & \cos 45°\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

$$\mathbf{M}^{-1}\mathbf{NM} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}\cos(-45°) & -\sin(-45°)\\ \sin(-45°) & \cos(-45°)\end{pmatrix}$$

$\mathbf{M}^{-1}\mathbf{NM}$ represents a rotation 45° clockwise about the origin.

19 a AB must be perpendicular to both O_1A and O_2B, since the line segment AB is tangential to the circles.

Then O_1O_2BA is a trapezium, with parallel side lengths 6 and 10, perpendicular base and oblique side length 30.



Considering the trapezium as a right-angled triangle O_1XO_2 atop a rectangle O_2XAB, the side lengths of the right triangle are $O_1X = 4, O_1O_2 = 30$

So angle $X\hat{O}_1O_2 = \cos^{-1}\left(\frac{4}{30}\right) = 82.3° = 1.44$ radians

b Considering the same triangle, $O_2X = \sqrt{30^2 - 4^2} = \sqrt{884} = AB = CD$

Then the bicycle chain consists of the large arc AD, the small arc BC and twice the length of the line segment AB

Since ABO_2O_1 is a trapezium with $O_1A \parallel O_2B$, $B\hat{O}_2O_1 = \pi - A\hat{O}_1O_2 = 1.70$

arc $AD = (2\pi - 2 \times 1.44) \times 10 = 34.1$ cm

arc $BC = (2\pi - 2 \times 1.70) \times 6 = 17.2$ cm

Total bike chain length $= 34.1 + 17.2 + 2\sqrt{884} = 111$ cm

20 a $OC = 4 = OB, BC = 3$

Cosine rule:

$$B\hat{O}C = \cos^{-1}\left(\frac{4^2 + 4^2 - 3^2}{2(4)(4)}\right) = 44.0° \text{ (0.769 radians)}$$

b Segment BC is the area of sector OBC less the area of triangle OBC. The shaded region is double that.

$$\begin{aligned}\text{Shaded area} &= r^2(\theta - \sin\theta)\\ &= 4^2(0.769 - \sin 0.769)\\ &= 1.18\ \text{cm}^2\end{aligned}$$

5 Functions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 5A

15 a $(\mathrm{f} \circ \mathrm{g})(x) = \mathrm{f}(\mathrm{g}(x)) = 3(4 - 3x) - 1 = 11 - 9x$

b $(\mathrm{g} \circ \mathrm{f})(x) = 4$

$4 - 3(3x - 1)) = 4$

$3(3x - 1) = 0$

$x = \dfrac{1}{3}$

16 a $\mathrm{f}(\mathrm{f}(x)) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 2$

b $\mathrm{f}(\mathrm{g}(x)) = (x - 1)^2 + 1 = x^2 - 2x + 2$

$\mathrm{g}(\mathrm{f}(x)) = (x^2 + 1) - 1 = x^2$

If these are equal then $x^2 = x^2 - 2x + 2$

$x = 1$

17 $\mathrm{f} \circ \mathrm{f}(x) = \mathrm{f}(\mathrm{f}(x)) = 2(2x^3)^3 = 16x^9$

18 a Restrict the domain so that the range of $g(x)$ lies entirely within the domain of $f(x)$

$3x + 10 \geq 4$

$x \geq -2$

b $\mathrm{f}(\mathrm{g}(x)) = 5$

$\sqrt{\mathrm{g}(x) - 4} = 5$

$\mathrm{g}(x) = 29$

$3x + 10 = 29$

$x = \dfrac{19}{3}$

19 a $(\mathrm{f} \circ \mathrm{g})(8) = \ln(8 - 5) = \ln 3$

$(\mathrm{g} \circ \mathrm{f})(8) = \ln 8 - 5$

b $(\mathrm{f} \circ \mathrm{g})(x) = \ln(x - 5) = 8$

$x = 5 + \mathrm{e}^8$

20 a i $(f \circ g)(3) = f(4) = 2$

ii $(g \circ f)(4) = g(2) = 1$

b $(f \circ g)(x) = 1$

$g(x) = 5$

$x = 4$

21 a i $(f \circ g)(3) = f(7) = 1$

ii $(g \circ f)(9) = g(9) = 1$

b $(f \circ g)(x) = 5$

$g(x) = 5$

$x = 1$

22 a Restrict the domain so that the range of $g(x)$ lies entirely within the domain of $f(x)$

$x - 3 > 0$

$x > 3$

b $(f \circ g)(x) = 1 = \ln(g(x))$

$g(x) = e = x - 3$

$x = e + 3$

c $(g \circ f)(x) = 1 = f(x) - 3$

$f(x) = 4 = \ln x$

$x = e^4$

d $(f \circ g)(x) = (g \circ f)(x)$

$\ln(x - 3) = \ln x - 3$

$\ln\left(\frac{x-3}{x}\right) = -3$

$\frac{x-3}{x} = e^{-3}$

$x(1 - e^{-3}) = 3$

$x = \frac{3}{1 - e^{-3}} = \frac{3e^3}{e^3 - 1}$

23 a

$$\begin{aligned}(f \circ g)(x) = f\left(\frac{1}{x-3}\right) &= \frac{1}{\frac{1}{x-3} + 2} \\ &= \frac{x-3}{1 + 2(x-3)}, x \neq 3 \\ &= \frac{x-3}{2x-5}, x \neq 3\end{aligned}$$

b Domain of $\mathrm{f}(x)$ is $x \neq -2$, domain of $\mathrm{g}(x)$ is $x \neq 3$

Restrict the domain so that the range of $g(x)$ lies entirely within the domain of $f(x)$

Domain of $(\mathrm{f} \circ \mathrm{g})\,(x)$ is $x \neq 3, \frac{5}{2}$

c

$$\frac{x-3}{2x-5} = 2$$

$$x - 3 = 4x - 10$$

$$x = \frac{7}{3}$$

24

$$(\mathrm{g} \circ \mathrm{f})\,(x) = 2\left(\frac{4}{x}\right)^2 = \frac{32}{x^2}$$

$$(\mathrm{f} \circ \mathrm{g})\,(x) = \frac{4}{2x^2} = \frac{2}{x^2}$$

So $(\mathrm{g} \circ \mathrm{f})\,(x) = 16\,(\mathrm{f} \circ \mathrm{g})\,(x)$

$k = 16$

25 a $\mathrm{f}(\mathrm{f}(x)) = \frac{1}{\frac{3}{3x+2}+2} = \frac{3x+2}{3+2(3x+2)}, x \neq -\frac{2}{3}$

$$= \frac{3x+2}{7+6x}, x \neq -\frac{2}{3}$$

Domain of $(\mathrm{f} \circ \mathrm{f})\,(x)$ is $x \neq -\frac{2}{3}, -\frac{7}{6}$

b $y \neq 0, \frac{1}{2}$

c

$$\frac{3x+2}{7+6x} = 1$$

$$3x + 2 = 7 + 6x$$

$$3x = -5$$

$$x = -\frac{5}{3}$$

Exercise 5B

18 a

$$\frac{4}{3-x} = 3$$

$$4 = 9 - 3x$$

$$x = \frac{5}{3}$$

b Let $y = \mathrm{f}(x) = \frac{4}{3-x}$ (range $\mathrm{f}(x) \neq 0$)

$y(3-x) = 4$

$xy = 3y - 4$

$x = \dfrac{3y-4}{y} = \mathrm{f}^{-1}(y)$

Changing variables:

$\mathrm{f}^{-1}(x) = \dfrac{3x-4}{x}$ (domain $x \neq 0$)

19 Let $y = \mathrm{f}(x) = 3\mathrm{e}^{5x}$ (range $\mathrm{f}(x) > 0$)

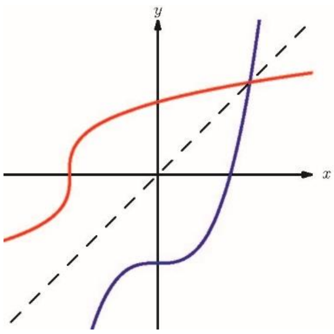
$\ln\left(\dfrac{y}{3}\right) = 5x$

$x = \dfrac{1}{5}\ln\left(\dfrac{y}{3}\right) = \mathrm{f}^{-1}(y)$

Change variables:

$\mathrm{f}^{-1}(x) = \dfrac{1}{5}\ln\left(\dfrac{x}{3}\right)$ (domain $x > 0$)

20 a The graph of the inverse is the graph of the original function reflected through $y = x$.

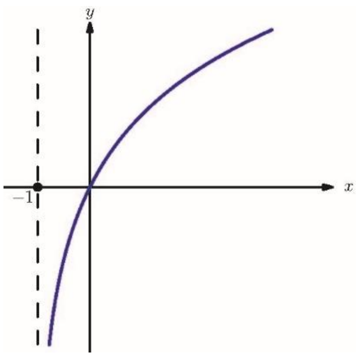


b Let $y = \mathrm{f}(x) = \frac{1}{5}x^3 - 3$ (range $\mathrm{f}(x) \in \mathbb{R}$)

Then $\mathrm{f}^{-1}(y) = x = \sqrt[3]{5(y+3)}$

Changing variables: $\mathrm{f}^{-1}(x) = \sqrt[3]{5x+15}$

21 a The graph of the inverse is the graph of the original function reflected through $y = x$.



b Let $y = \mathrm{f}(x) = \mathrm{e}^{\frac{x}{2}} - 1$ (range $\mathrm{f}(x) > -1$)

Then $\mathrm{f}^{-1}(y) = x = 2\ln(y+1)$

Changing variables:

$f^{-1}(x) = 2\ln(x+1)$ (domain $x > -1$)

22 **a** Let $y = f(x) = \frac{x^2+1}{x^2-4} = 1 + \frac{5}{x^2-4}$ (domain $x > 2$, range $f(x) > 1$)

$$y - 1 = \frac{5}{x^2 - 4}$$

$$x^2 - 4 = \frac{5}{y-1}$$

$$x = \sqrt{4 + \frac{5}{y-1}} = \sqrt{\frac{4y+1}{y-1}} = f^{-1}(y)$$

Changing variables:

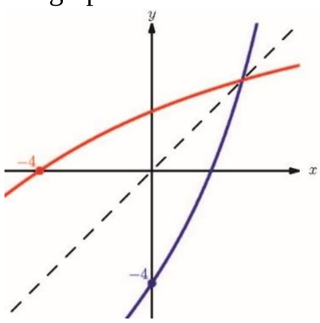
$$f^{-1}(x) = \sqrt{\frac{4x+1}{x-1}} \text{ (domain } x > 1\text{)}$$

b Range $f^{-1}(x) > 2$

23 $(g^{-1} \circ f^{-1})(x) = 4$ so

$$\begin{aligned} x &= (f \circ g)(4) \\ &= f\big(g(4)\big) \\ &= \frac{1}{4 + \sqrt{2 \times 4 + 1}} \\ &= \frac{1}{7} \end{aligned}$$

24 **a** The graph of the inverse is the graph of the original function reflected through $y = x$.



b From the graph is it clear that the intersection of $y = f(x)$ and $y = f^{-1}(x)$ is also the intersection with $y = x$

$$f(x) = x$$

$$e^{\frac{x}{2}} + x - 5 = x$$

$$e^{\frac{x}{2}} = 5$$

$$x = 2\ln 5$$

25 **a** Since $f(x) = f(-x)$, $f(x)$ is one-to-one for $x \le 0$ so the largest possible value is $a = 0$

b $f(x) = x^2 + 3$ for $x \le 0$ has range $f(x) \ge 3$

Let $y = f(x)$ so $f^{-1}(y) = x = -\sqrt{y-3}$ for $y \ge 3$

Changing variables: $f^{-1}(x) = -\sqrt{x-3}$ for $x \geq 3$.

26 a Since $g(5+x) = g(5-x)$, $g(x)$ is one-to-one for $x \leq 5$ so the largest possible value is $k = 5$.

b $g(x) = 9(x-5)^2$ for $x \leq 5$ has range $g(x) \geq 0$

Let $y = g(x)$ so $g^{-1}(y) = x = 5 - \frac{\sqrt{y}}{3}$ for $y \geq 0$

Changing variables:

$$g^{-1}(x) = 5 - \frac{1}{3}\sqrt{x} \text{ for } x \geq 0$$

27 $f(x) = \frac{(ax+3)}{x-4}$ has domain $x \neq 4$

If $f(x) \equiv f^{-1}(x)$ then $(f \circ f)(x) \equiv x$.

$$(f \circ f)(x) = \frac{a\left(\frac{ax+3}{x-4}\right) + 3}{\frac{ax+3}{x-4} - 4}$$

Multiplying numerator and denominator by $(x-4)$ to simplify:

$$\begin{aligned}(f \circ f)(x) &= \frac{a(ax+3) + 3(x-4)}{(ax+3) - 4(x-4)} \\ &= \frac{x(a^2+3) + 3a - 12}{(a-4)x + 19} \equiv x\end{aligned}$$

$$x(a^2+3) + 3a - 12 \equiv x^2(a-4) + 19x$$

Comparing coefficients: If $a = 4$ then the x^2 term disappears, as is necessary, and the equivalence is met:

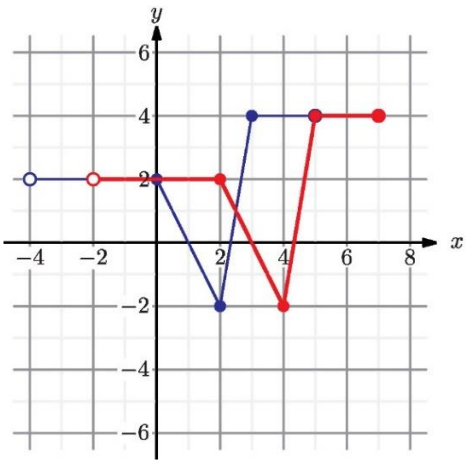
$$19x + 0 \equiv 0x^2 + 19x$$

So for $f(x) \equiv f^{-1}(x)$, $a = 4$

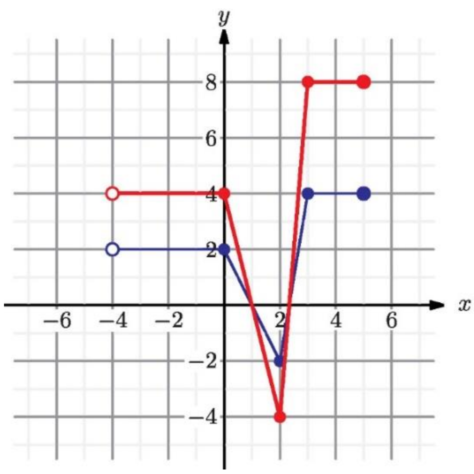
Comment: An alternative approach to a solution involves finding $f^{-1}(x)$ in terms of a and then requiring the two function expressions to be equivalent.

Exercise 5C

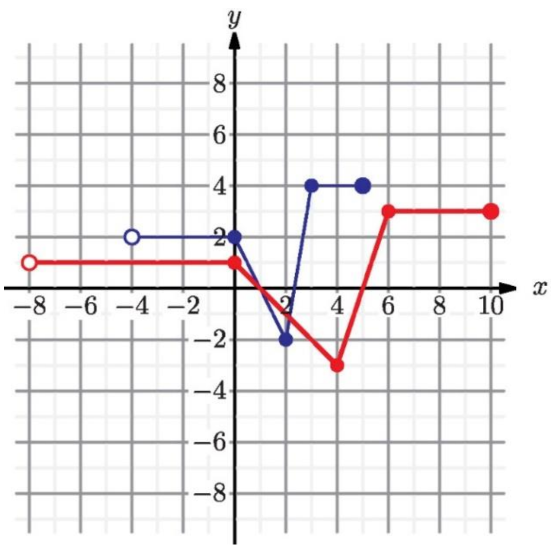
31 a $\mathrm{f}(x-2)$ is $\mathrm{f}(x)$ with a translation $\begin{pmatrix}2\\0\end{pmatrix}$:



b $2\mathrm{f}(x)$ is $\mathrm{f}(x)$ after a vertical stretch with scale factor 2:



c $y=\mathrm{f}\left(\frac{1}{2}x\right)-1$ is $\mathrm{f}(x)$ after a horizontal stretch with scale factor 2 and a translation $\begin{pmatrix}0\\-1\end{pmatrix}$:



32 $y_1 = 3x^2 - 4x$

Translation $\begin{pmatrix}-3\\0\end{pmatrix}$: Replace x with $(x+3)$

$$\begin{aligned} y_2 &= 3(x+3)^2 - 4(x+3) \\ &= 3x^2 + 14x + 15 \end{aligned}$$

Vertical stretch scale factor 4: $y_3 = 4y_2$

$y_3 = 12x^2 + 56x + 60$

33 $y_1 = \mathrm{e}^x$

Translation $\begin{pmatrix}2\\0\end{pmatrix}$: Replace x with $(x-2)$

$y_2 = \mathrm{e}^{x-2}$

Vertical stretch scale factor 3: $y_3 = 3y_2$

$y_3 = 3\mathrm{e}^{x-2}$

34 **a** $x^2 - 10x + 11 = (x-5)^2 - 14$

b $y_1 = x^2$

Translation $\begin{pmatrix}5\\0\end{pmatrix}$Replace x with $(x-5)$:

$y_2 = (x-5)^2$

Translation $\begin{pmatrix}0\\-14\end{pmatrix}$: $y_3 = y_2 - 14$

$y_3 = (x-5)^2 - 14$

A translation $\begin{pmatrix}5\\-14\end{pmatrix}$ transforms $y = x^2$ to $y = x^2 - 10x + 11$

35 **a** $5x^2 + 30x + 45 = 5(x^2 + 6x + 9) = 5(x+3)^2$

b $y_1 = x^2$

Translation $\begin{pmatrix}-3\\0\end{pmatrix}$Replace x with $(x+3)$:

$y_2 = (x+3)^2$

Vertical stretch with scale factor 5: $y_3 = 5y_2$

$y_3 = 5(x+3)^2$

A translation $\begin{pmatrix}-3\\0\end{pmatrix}$ and a vertical stretch with scale factor 5 transforms $y = x^2$ to $y = 5x^2 + 30x + 45$.

36 a $y_1 = x^2 = \mathrm{f}(x)$

Vertical stretch with scale factor 9: $y_2 = 9y_1$

$y_2 = 9x^2$

b $y_2 = (\pm 3x)^2 = \mathrm{f}(\pm 3x)$

From y_1 to y_2: Replace x with $3x$: Horizontal stretch with scale factor $\frac{1}{3}$

So to reverse the transformation would require a horizontal stretch with scale factor 3.

Comment: A stretch with scale factor $-\frac{1}{3}$ would cause x to be replaced by $-3x$ as an alternative solution, but this would normally be interpreted as two separate transformations: a stretch and a reflection through the y-axis

37 a $y_1 = 2x^3 = \mathrm{f}(x)$

Vertical stretch with scale factor 8: $y_2 = 8y_1$

$y_2 = 16x^3$

b $y_2 = 2(2x)^3 = \mathrm{f}(2x)$

Replace x with $2x$: Horizontal stretch with scale factor $\frac{1}{2}$

38 $y_1 = 2x^3 - 5x^2$

Reflection through x-axis: $y_2 = -y_1$

$y_2 = -2x^3 + 5x^2$

Reflection through y-axis: Replace x with $-x$

$y_3 = 2x^3 + 5x^2$

39 a $y_1 = \sqrt{x}$

Translation $\begin{pmatrix}1\\0\end{pmatrix}$: Replace x with $(x-1)$

$y_2 = \sqrt{x-1}$

Horizontal stretch with scale factor $\frac{1}{2}$: Replace x with $2x$

$y_3 = \sqrt{2x-1}$

A translation $\begin{pmatrix}1\\0\end{pmatrix}$ followed by a horizontal stretch with scale factor $\frac{1}{2}$ transforms $y = \sqrt{x}$ to $y = \sqrt{2x-1}$

b $y_1 = \sqrt{x}$

Horizontal stretch with scale factor $\frac{1}{2}$: Replace x with $2x$

$y_2 = \sqrt{2x}$

Translation $\begin{pmatrix}1\\0\end{pmatrix}$: Replace x with $(x-1)$

$y_3 = \sqrt{2(x-1)}$

40 a $y_1 = ax^2 + bx + c$

Translation $\begin{pmatrix}2\\0\end{pmatrix}$: Replace x with $(x-2)$

$$\begin{aligned} y_2 &= a(x-2)^2 + b(x-2) + c \\ &= ax^2 + (b-4a)x + 4a - 2b + c \end{aligned}$$

Horizontal stretch with scale factor 3: Replace x with $\frac{x}{3}$

$$\begin{aligned} y_3 &= \frac{a}{9}x^2 + \frac{(b-4a)}{3}x + 4a - 2b + c \\ &= x^2 + cx + 14 \end{aligned}$$

Comparing coefficients:

$$\begin{cases} x^2: \dfrac{a}{9} = 1 \Rightarrow a = 9 & \\ x^1: \dfrac{b-4a}{3} = c \Rightarrow b - 36 = 3c & (1) \\ x^0: 4a - 2b + c = 14 \Rightarrow 22 - 2b + c = 0 & (2) \end{cases}$$

$(2) + 2(1): 22 + c - 72 = 6c$

$5c = -50$

$c = -10$

$(1): b = 36 + 3c = 6$

$a = 9, b = 6, c = -10$

41 $y_1 = \mathrm{f}(x)$

Translation $\begin{pmatrix} -8 \\ 0 \end{pmatrix}$: Replace x with $(x+8)$

$y_2 = \mathrm{f}(x+8)$

Horizontal stretch with scale factor 0.25: Replace x with $4x$

$y_3 = f(4x+8)$

Alternatively:

$y_1 = \mathrm{f}(x)$

Horizontal stretch with scale factor 0.25: Replace x with $4x$

$y_2 = \mathrm{f}(4x)$

Translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$: Replace x with $(x+2)$

$y_3 = f\big(4(x+2)\big) = \mathrm{f}(4x+8)$

Either a translation $\begin{pmatrix} -8 \\ 0 \end{pmatrix}$ followed by a horizontal stretch with scale factor 0.25

Or a horizontal stretch with scale factor 0.25 followed by a translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

42 **a** $y_1 = \mathrm{f}(x)$

Vertical stretch with scale factor 5: $y_2 = 5y_1$

$y_2 = 5\mathrm{f}(x)$

Translation $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$: $y_3 = y_2 + 3$

$y_3 = 5\mathrm{f}(x) + 3$

b $y_3 = 5\left(\mathrm{f}(x) + \frac{3}{5}\right)$

Translation $\begin{pmatrix} 0 \\ 0.6 \end{pmatrix}$ followed by a vertical stretch with scale factor 5.

43 A stretch from the line $y = 1$ is equivalent to

Translation $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Stretch from the line $y = 0$

Translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$y_1 = \mathrm{f}(x)$

After (1):

$y_2 = \mathrm{f}(x) - 1$

After (2):

$y_3 = 2(\mathrm{f}(x) - 1) = 2\mathrm{f}(x) - 2$

After (3):

$y_4 = (2\mathrm{f}(x) - 2) + 1 = 2\mathrm{f}(x) - 1$

Exercise 5D

19 $m = 1.2\mathrm{e}^{-0.26t}$

a $m(0) = 1.2$

$m(t) = 1.2\mathrm{e}^{-0.26t} < 0.6$

$$t > -\frac{1}{0.26}\ln(0.5) \approx 2.67$$

After 3 days, the mass has fallen below half its initial value

b $m(t) = 1.2\mathrm{e}^{-0.26t} < 0.1$

$$t > -\frac{1}{0.26}\ln\left(\frac{1}{12}\right) \approx 9.56$$

After 10 days, the mass has fallen to below 0.1 mg

20 a $T = A\mathrm{e}^{-kt}$

$T(0) = A = 23$

$T(15) = 23\mathrm{e}^{-15k} = 11.5$

$$k = \frac{1}{15}\ln 2 = 0.0462$$

b $T(t) = 8 = 23\mathrm{e}^{-kt}$

$$t = -\frac{1}{k}\ln\left(\frac{8}{23}\right) \approx 22.9$$

It takes 22.9 minutes to cool to 8 °C.

c It is assumed that the bowl of ice remains at a constant 0 °C, and is not warming significantly as the ice cream cools.

21 a $R = p \ln A + q$

$$\begin{cases} 3 = p \ln 2 + q & (1) \\ 4 = p \ln 25 + q & (2) \end{cases}$$

$(2) - (1): 1 = p \ln 12.5$

$$p = \frac{1}{\ln 12.5} \approx 0.396$$

$q = 3 - p \ln 2 \approx 2.73$

b $R(100) = p \ln 100 + q \approx 4.55$

c $A = e^{\frac{R-q}{p}}$

When $R = 2$, $A = 0.16$

When $R = 5$, $A = 312.5$

22 a $V(4) = 100 = 4p \Rightarrow p = 25$

b The volume must be continuous in time, so $V(5) = 5p = 25q \Rightarrow q = 5$

c When $V = 4500, t = \sqrt{\frac{4500}{q}} = 30$ seconds.

23 $P(0) = A = P_0$

$$P(3) = Ae^{3k} = 2P_0 \Rightarrow k = \frac{1}{3}\ln 2 \approx 0.231$$

$P = P_0 e^{0.231t}$

24 a When t increases by 0.5, the value of S would halve, so the halving time is 5 years according to the model.

b $4^t = \frac{400}{S} \Rightarrow t = \frac{\ln\left(\frac{400}{S}\right)}{\ln 4}$

When $S = 1$, $t = \frac{\ln 400}{\ln 4} \approx 4.32$ so it is projected that semiconductor chips will reach 1 micron size approximately 43.2 years after their invention.

25 a $T = B + Ae^{-\mathrm{kt}}$

$T(0) = 90 = A + B$

$T(t) \to 20 = B$ as $t \to \infty$

$A = 70, B = 20$

b $T(4) = 60 = 20 + 70e^{-4k}$

$$k = -\frac{1}{4}\ln\left(\frac{40}{70}\right) \approx 0.140$$

c $t = -\frac{1}{k}\ln\left(\frac{T-B}{A}\right)$

When $T = 50$, $t \approx 6.06$ minutes

26 a The oven is at 200 °C

The initial temperature difference is 178 °C

b $T(0) = 22 = 200 - A \Rightarrow A = 178$

c $T(4) = 111 = 200 - 178e^{-4k}$

$$k = -\frac{1}{4}\ln\left(\frac{89}{178}\right) \approx 0.173$$

d

$$t = -\frac{1}{k}\ln\left(\frac{200 - T}{178}\right)$$

When $T = 180$, $t = 12.6$ minutes

27 a If it takes 10 years to quarter the mass then the half life is 5 years

b

$$M(5) = \frac{M_0}{2} = M_0e^{-5k} \Rightarrow k = -\frac{1}{5}\ln\left(\frac{1}{2}\right) \approx 0.139$$

c

$$M(20) = \frac{1}{16}M_0 = 0.6 \Rightarrow M_0 = 9.6\text{ g}$$

28 a $P_A = a\sin bt + d$

Period is 24 hours $= \frac{2\pi}{b} \Rightarrow b = \frac{\pi}{12}$

Amplitude a is the half the difference between maximum and minimum; $a = 20$

Midlevel d is the mean of maximum and minimum; $d = 30$

$$P_A = 20\sin\left(\frac{\pi t}{12}\right) + 30$$

b $P_P = A\sin B(t - C) + D$

The period is still 24 hours so $B = \frac{\pi}{12}$

Maximum for P_A occurs when $t = 6$ so the maximum for P_P must occur at $t = 10$ and $C = 4$

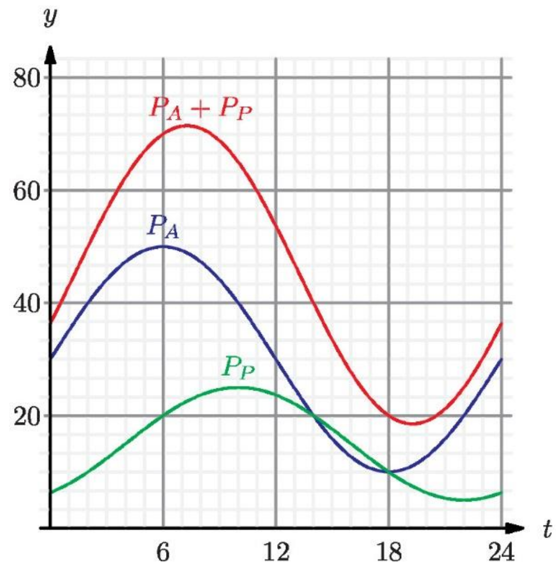
Amplitude A is the half the difference between maximum and minimum; $A = 10$

Midlevel D is the mean of maximum and minimum; $D = 15$

$$P_P = 10\sin\left(\frac{\pi}{12}(t - 4)\right) + 15$$

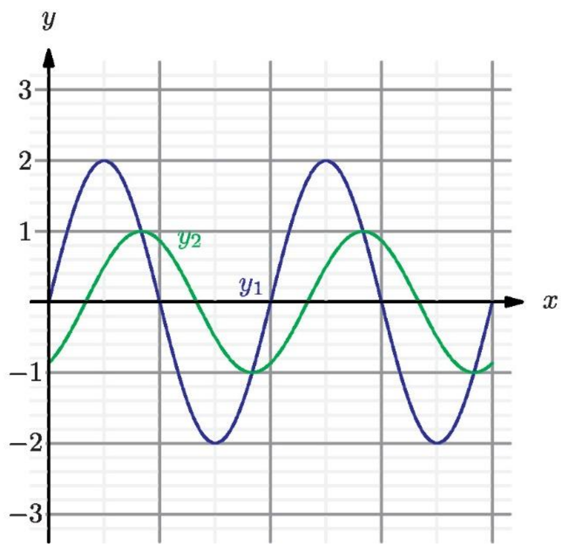
c From the GDC, the first intersection of the two curves is at $t = 14$ hours

d



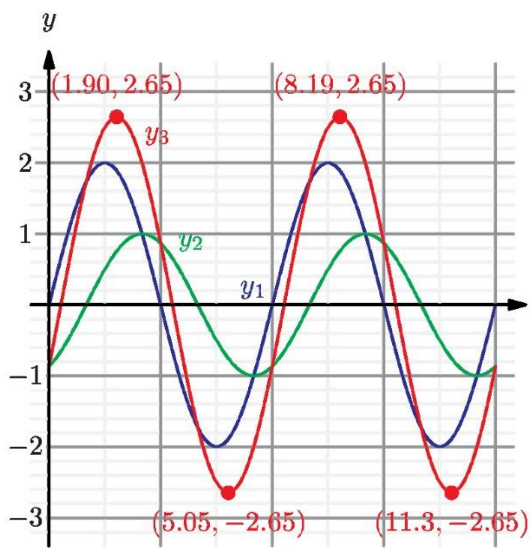
y
80
60
40
20
6
12
18
24
t
$P_A + P_P$
P_A
P_P

29 a



y
3
2
1
−1
−2
−3
x
y_2
y_1

b



y
3
2
1
−1
−2
−3
x
(1.90, 2.65)
(8.19, 2.65)
y_3
y_2
y_1
(5.05, −2.65)
(11.3, −2.65)

c $y_3 = a \sin b(x - c)$

The sum has the same period of 2π so $b = 1$

The amplitude is now 2.65 from the graph so $a = 2.65$

The maximum occurs at $1.90 = \frac{\pi}{2} + 0.333$ so $c = 0.333$

$y_3 = 2.65 \sin(x - 0.333)$

30 a Assume $k > 0$, without which the population would decrease to zero.

As $t \to \infty$, $N \to L$ so $L = 500$

$$N(2) = 300 = \frac{500}{1 + Ce^{-2k}}$$

$$1 + Ce^{-2k} = \frac{5}{3}$$

$$C = \frac{2}{3}e^{2k}(1)$$

b $C(5) = 400 = \frac{500}{1+Ce^{-5k}}$

$$1 + Ce^{-5k} = \frac{5}{4}$$

$$C = \frac{1}{4}e^{5k}(2)$$

c

$$(2) \div (1): \frac{3}{8}e^{3k} = 1$$

$$k = \frac{1}{3}\ln\left(\frac{8}{3}\right) = 0.327$$

$$C = \frac{2}{3}e^{2k} = 1.28$$

d $N(0) = \frac{500}{1+C} = 219$

e $N(t) \geq 480$

$$\frac{500}{1 + Ce^{-kt}} \geq 480$$

$$1 + Ce^{-kt} \leq \frac{500}{480}$$

$$Ce^{-kt} \leq \frac{1}{24}$$

$$-kt \leq \ln\left(\frac{1}{24C}\right)$$

$$t \geq \frac{1}{k}\ln(24C) \approx 10.5$$

Require 11 years to exceed 480 birds.

31 a Assume that the initial population before introduction was zero and that there is no rabbit death or breeding to affect the population.

On that basis, the population after three years will be 300

b If the population is $P(t)$ for time t years after the introduction ceased (so that $P(t-3)$ is the population for time $t$years after introduction began):

$$P = \frac{L}{1 + Ce^{-kt}}$$

$$\begin{cases} P(0) = 300 = \dfrac{L}{1+C} & (1) \\ P(1) = 450 = \dfrac{L}{1+Ce^{-k}} & (2) \\ P \to 1000 = L \text{ as } t \to \infty & (3) \end{cases}$$

$(3)\colon L = 1000$

$$(1)\colon C = \left(\frac{L}{300}\right) - 1 = \frac{7}{3}$$

$$(2)\colon k = -\ln\left(\frac{1}{C}\left(\frac{L}{450} - 1\right)\right) \approx 0.647$$

Then under the model, $P(2) \approx 610$

32 a $N = \frac{L}{1+Ce^{-kt}}$

L is the carrying capacity, 2000

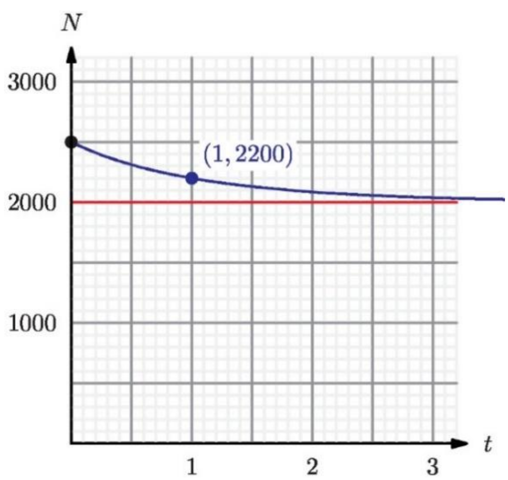
$$N(0) = 2500 = \frac{L}{1+C}$$

$$C = \frac{L}{2500} - 1 = -0.2$$

b $N(1) = 2200 = \frac{L}{1+Ce^{-k}}$

$$k = -\ln\left(\frac{\frac{L}{2200} - 1}{C}\right) \approx 0.8 \text{ (1DP)}$$

c



33 a For the first period, with growth, $N = N_0 e^{\alpha t}$

$N(0.5) = N_0 e^{0.5\alpha}$

Then the second period of decay must be continuous at $t = 0.5$ so

$N = (N_0 e^{0.5\alpha}) e^{-\beta(t-0.5)}$

$$N(t) = \begin{cases} N_0 e^{\alpha t} & 0 \le t \le 0.5 \\ (N_0 e^{0.5\alpha}) e^{-\beta(t-0.5)} & 0.5 \le t \le 1 \end{cases}$$

b The population will continue to increase, on average, if the population at $t = 1$ is greater than N_0.

$N_0 e^{0.5(\alpha-\beta)}$

Require that $\alpha - \beta > 0$

$\alpha > \beta$

34 a $N = 200e^{-kt}$ where $k = \frac{\ln 2}{2} \approx 0.347$

b Since N represents the number of individual tigers, $N < 1$ would mean there were no tigers remaining and that the population has become extinct.

c $t = -\frac{1}{k} \ln\left(\frac{N}{200}\right)$

When $N = 1, t \approx 15.3$

The tigers will go extinct after approximately 15.3 years.

d A population of 200 is already sufficiently small that a continuous model becomes unreliable; when the population shrinks further, individual events governing death or birth of tigers will perturb the model to an increasing extent. It is clear that when the population reaches a critical low number, reproduction would be compromised or – depending on the balance of gender in the population – may become impossible.

35 a $T = a + be^{-kt}$

$$\begin{cases} T \to 20 = a \text{ as } t \to \infty & (1) \\ T(0) = 100 = a + b & (2) \\ T(q) = 30 = a + be^{-qk} & (3) \end{cases}$$

$(1): a = 20$

$(2): b = 80$

$$(3): k = -\frac{1}{q} \ln\left(\frac{30-a}{b}\right) = -\frac{1}{q} \ln\left(\frac{1}{8}\right) = \frac{3\ln 2}{q}$$

The model is $T = 20 + 80e^{-\frac{3\ln 2}{q}t}$

b Rearranging: $-\frac{3\ln 2}{q} t = \ln\left(\frac{T-20}{80}\right)$ so $t = \frac{q\ln\left(\frac{80}{T-20}\right)}{3\ln 2}$

When $T = 50$:

$$t = \frac{q\ln\left(\frac{8}{3}\right)}{3\ln 2}$$

Mixed Practice 5

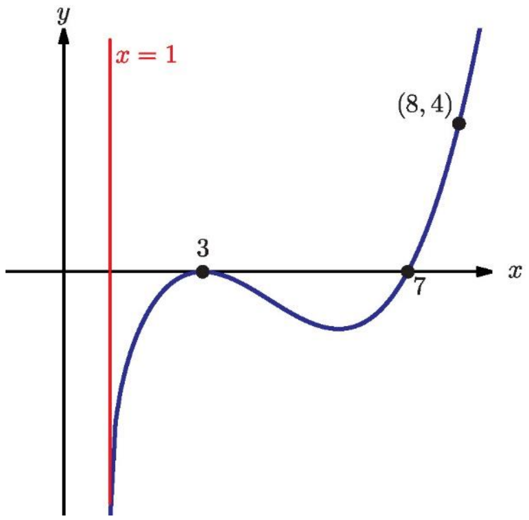
1 **a** $y = \mathrm{f}(x)$

$y_1 = \mathrm{f}(x-3)$: Translation $\begin{pmatrix}3\\0\end{pmatrix}$

Asymptote $x = -2$ moves to $x = 1$, $(0,0)$ moves to $(3,0)$, $(4,0)$ moves to $(7,0)$, $(5,2)$ moves to $(8,2)$

$y_2 = 2\mathrm{f}(x-3)$: Vertical stretch, scale factor 2

Asymptote $x = 1$, $(3,0)$ and $(7,0)$ unaffected, $(8,2)$ moves to $(8,4)$.



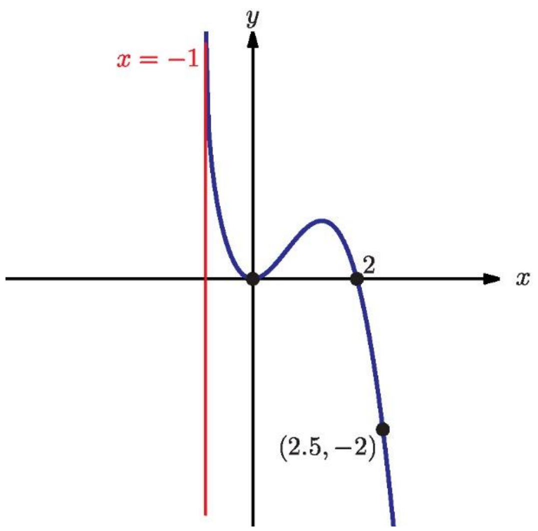
b $y = \mathrm{f}(x)$

$y_1 = \mathrm{f}(2x)$: Horizontal stretch scale factor $\frac{1}{2}$

Asymptote $x = -2$ moves to $x = -1$,$(0,0)$ unaffected, $(4,0)$ moves to $(2,0)$, $(5,2)$ moves to $(2.5,2)$

$y_2 = -\mathrm{f}(2x)$: Reflection through x-axis

Asymptote $x = -1$,$(0,0)$ and $(2,0)$ unaffected, $(2.5,2)$ moves to $(2.5,-2)$.



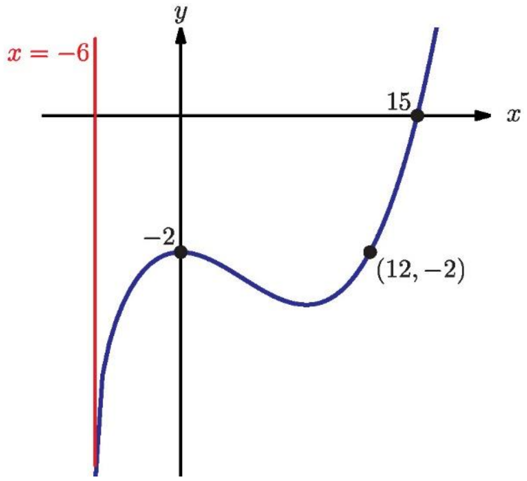
c $y = \mathrm{f}(x)$

$y_1 = \mathrm{f}\left(\frac{x}{3}\right)$: Horizontal stretch scale factor 3

Asymptote $x = -2$ moves to $x = -6$, (0,0) unaffected, (4,0) moves to (12,0), (5,2) moves to (15,2)

$y_2 = \mathrm{f}\left(\frac{x}{3}\right) - 2$: Translation $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

Asymptote $x = -6$ unaffected, $(0,0)$ moves to $(0,-2)$, $(12,0)$ moves to $(12,-2)$, $(15,2)$ moves to $(15,0)$.



2 a Let $y = \mathrm{f}(x) = 3x - 1$

Then $y + 1 = 3x$

So $x = \mathrm{f}^{-1}(y) = \frac{1}{3}(y+1)$

Changing variables: $\mathrm{f}^{-1}(x) = \frac{1}{3}(x+1)$

b

$$\begin{aligned}(\mathrm{f} \circ \mathrm{f}^{-1})(x) &= \mathrm{f}(\mathrm{f}^{-1}(x)) \\ &= \mathrm{f}\left(\frac{1}{3}(x+1)\right) \\ &= 3\left(\frac{1}{3}(x+1)\right) - 1 \\ &= x\end{aligned}$$

3 a Square root must take non-negative arguments, so the greatest possible domain of $h(x)$ is $x \leq 5$.

$a = 5$.

b

$\sqrt{5-x} = 3$

$5 - x = 9$

$x = -4$

4 $\mathrm{e}^{3x} = 4$ so $x = \frac{1}{3}\ln 4 = 0.462$

5 a

$$\begin{aligned}(\mathrm{f}\circ\mathrm{f})(2) &= \mathrm{f}(\mathrm{f}(2))\\ &= \mathrm{f}(0)\\ &= 3\end{aligned}$$

b $\mathrm{f}^{-1}(4) = 1$

6

$$\begin{aligned}(\mathrm{g}\circ\mathrm{f})(x) &= \mathrm{g}(\mathrm{f}(x))\\ &= \mathrm{e}^{2(x+3)} = 1\end{aligned}$$

$2(x+3) = \ln 1 = 0$

$x = -3$

7 a $\ln x$ takes only positive arguments for range in $\mathbb{R}$ so the domain of $g(x)$ is restricted to $x > 2$

b Let $y = \mathrm{g}(x) = 3\ln(x-2)$ with domain $x > 2$ and range $\mathbb{R}$

$\frac{y}{3} = \ln(x-2)$

$x = \mathrm{g}^{-1}(y) = 2 + \mathrm{e}^{\frac{y}{3}}$ with domain $x > 2$ and range $\mathbb{R}$

Changing variables: $\mathrm{g}^{-1}(x) = 2 + \mathrm{e}^{\frac{x}{3}}$ with domain $\mathbb{R}$ and range $\mathrm{g}^{-1}(x) > 2$.

8 By inspection, $\mathrm{g}^{-1}(x) = \sqrt[3]{x}$ and $\mathrm{f}^{-1}(x) = \frac{x-1}{3}$.

$$\begin{aligned}(\mathrm{f}\circ\mathrm{g})^{-1}(x) &= (\mathrm{g}^{-1}\circ\mathrm{f}^{-1})(x)\\ &= \sqrt[3]{\frac{x-1}{3}}\end{aligned}$$

9 a

$$\begin{aligned}(\mathrm{f}\circ\mathrm{g})(x) &= \mathrm{f}(\mathrm{g}(x))\\ &= 2(x^3) + 3\\ &= 2x^3 + 3\end{aligned}$$

b $2x^3 + 3 = 0$ so $x = \sqrt[3]{-\frac{3}{2}} = -1.14$ (3sf)

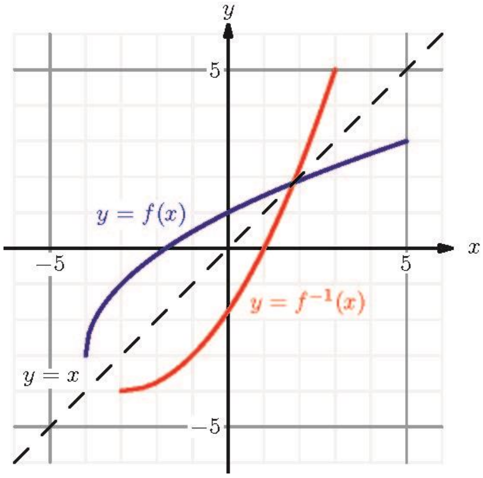
10 a From the graph:

i $\mathrm{f}(-3) = -1$

ii $\mathrm{f}^{-1}(1) = 0$

b The domain of f^{-1} is the range of f: $[-3,3]$

c The graph of f^{-1} is the graph of f under a reflection through the line $y = x$.



11 a

$$\begin{aligned}(\mathrm{f} \circ \mathrm{g})(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \frac{1}{x-1} - 2\end{aligned}$$

$$\frac{1}{x-1}$$

The domain of $\mathrm{g}(x)$ is $x \neq 1$ and the domain of $\mathrm{f}(x)$ is unrestricted so the domain of $(\mathrm{f} \circ \mathrm{g})(x)$ is $x \neq 1$.

b Let $y = (\mathrm{f} \circ \mathrm{g})(x) = \frac{1}{x-1} - 2$

Then $y + 2 = \frac{1}{x-1}$

Taking reciprocals on both sides:$x - 1 = \frac{1}{y+2}$

Then $x = (\mathrm{f} \circ \mathrm{g})^{-1}(y) = 1 + \frac{1}{y+2}$

Changing variables: $(\mathrm{f} \circ \mathrm{g})^{-1}(x) = 1 + \frac{1}{x+2}$

By inspection, $\mathrm{f}^{-1}(x) = x + 2$ and $\mathrm{g}^{-1}(x) = 1 + \frac{1}{x}$

$$\begin{aligned}(\mathrm{g}^{-1} \circ \mathrm{f}^{-1})(x) &= \mathrm{g}^{-1}(\mathrm{f}^{-1}(x)) \\ &= 1 + \frac{1}{x+2}\end{aligned}$$

This demonstrates that $(\mathrm{f} \circ \mathrm{g})^{-1}(x) \equiv (\mathrm{g}^{-1} \circ \mathrm{f}^{-1})(x)$

12 Let $y = \mathrm{f}(x) = \frac{3x-1}{x+4}$, which has domain $x \neq -4$ and range $\mathrm{f}(x) \neq 3$

$y(x + 4) = 3x - 1$

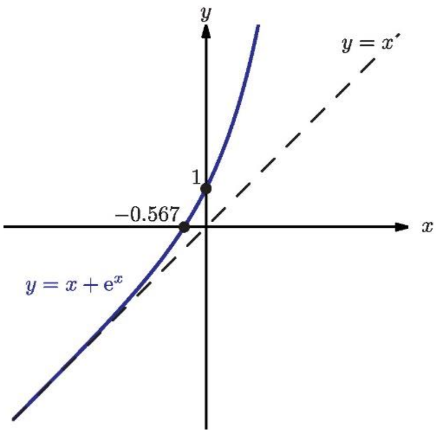
$xy - 3x = -4y - 1$

$x(3 - y) = 4y + 1$

$x = \mathrm{f}^{-1}(y) = \dfrac{4y + 1}{3 - y}$

Changing variables: $\mathrm{f}^{-1}(x) = \frac{4x+1}{3-x}$ with domain $x \neq 3$ and range $\mathrm{f}^{-1}(x) \neq -4$.

13 a



The graph shows that $\mathrm{g}(x)$ is one-to-one (for no value k are there two distinct values x_1 and x_2 such that $\mathrm{g}(x_1) = \mathrm{g}(x_2) = k$; this is seen in the graph as the quality that any horizontal line crosses the curve in at most one place). Therefore $\mathrm{g}^{-1}(x)$ exists. Since the range of $\mathrm{g}(x)$ is $\mathbb{R}$, the inverse function has domain $\mathbb{R}$.

b $\mathrm{g}^{-1}(x) = 2$ so $x = \mathrm{g}(2) = 2 + \mathrm{e}^2 = 9.39$

14 a

$$\begin{aligned}(\mathrm{g} \circ \mathrm{f})\left(\frac{1}{4}\right) &= \mathrm{g}\left(\mathrm{f}\left(\frac{1}{4}\right)\right)\\ &= \mathrm{g}\left(\frac{1}{2}\right)\\ &= 3\end{aligned}$$

b $(\mathrm{f}^{-1} \circ \mathrm{g})(x) = \frac{1}{3}$ so

$$\begin{aligned}x &= \mathrm{g}^{-1}\left(\mathrm{f}\left(\frac{1}{3}\right)\right)\\ &= \log_9\left(3^{-\frac{1}{2}}\right)\\ &= \log_9\left(9^{-\left(\frac{1}{4}\right)}\right)\\ &= -\frac{1}{4}\end{aligned}$$

15 a $\mathrm{h}(x) = \ln(x-2)$ fof $x > 2$ so $\mathrm{h}^{-1}(x) = 2 + \mathrm{e}^x$ with range $\mathrm{h}^{-1}(x) > 2$

b $(\mathrm{g} \circ \mathrm{h})(x) = \mathrm{e}^{\ln(x-2)} = x - 2$

16 a The graph is symmetrical about $x = -\frac{1}{2}$ so the graph is one-to-one for $x \leq -\frac{1}{2}$

$$a = -\frac{1}{2}$$

b Let $y = \mathrm{f}(x) = (2x+1)^2$

Then $x = \mathrm{f}^{-1}(y) = \frac{-\sqrt{y}-1}{2}$ (selecting the negative root since the domain of f is to the negative side of $x = -\frac{1}{2}$)

Changing variables: $\mathrm{f}^{-1}(x) = \frac{-\sqrt{x}-1}{2}$

17 a $\mathrm{f}(5) = 5^2 + 3 = 28$

b $\mathrm{gf}(x) = 12 - (x^2 + 3) = 9 - x^2$

i The graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are reflections through the line $y = x$.

ii Let $y = \mathrm{f}(x) = x^2 + 3$

Rearranging gives $x = \mathrm{f}^{-1}(y) = \sqrt{y - 3}$

Changing variables gives $\mathrm{f}^{-1}(x) = \sqrt{x - 3}$

iii The domain of $\mathrm{f}(x)$ is $x \geq 1$ and so the range of $\mathrm{f}^{-1}(x)$ is $\mathrm{f}^{-1}(x) \geq 1$

18 a i $\mathrm{f}(7) = 3(7) + 1 = 22$

ii

$$\begin{aligned}(\mathrm{f} \circ \mathrm{g})(x) &= \mathrm{f}\big(\mathrm{g}(x)\big)\\ &= \frac{3(x+4)}{x-1} + 1\\ &= \frac{(3x+12) + (x-1)}{x-1}\\ &= \frac{4x+11}{x-1}\end{aligned}$$

iii $(\mathrm{f} \circ \mathrm{f})(x) = 3(3x + 1) + 1 = 9x + 4$

b The domain of f is $\mathbb{R}$ and f is a linear transformation, so f has range $\mathbb{R}$.

Since the domain of g requires $x \neq 1$, $(\mathrm{g} \circ \mathrm{f})(x)$ is not well-defined across the whole domain of f.

c i

$$\begin{aligned}(\mathrm{g} \circ \mathrm{g})(x) &= \frac{\frac{x+4}{x-1} + 4}{\frac{x+4}{x-1} - 1}\\ &= \frac{x + 4 + 4(x-1)}{x + 4 - (x-1)}\\ &= \frac{5x}{5}\\ &= x\end{aligned}$$

Since $(\mathrm{g} \circ \mathrm{g})(x) \equiv x$, it follows that $\mathrm{g}(x) \equiv \mathrm{g}^{-1}(x)$

ii Since $\mathrm{g}(x)$ is self-inverse, the range and domain are the same, so the range of $\mathrm{g}(x)$ is $\mathrm{g}(x) \neq 1$.

19 a $\mathrm{f}(x) = \sqrt{x - 5}$ so $\mathrm{f}^{-1}(x) = x^2 + 5$

$\mathrm{f}^{-1}(2) = 9$

b

$$\begin{aligned}(\mathrm{f} \circ \mathrm{g}^{-1})(3) &= \mathrm{f}\big(\mathrm{g}^{-1}(3)\big)\\ &= \mathrm{f}(30)\\ &= \sqrt{30 - 5} = 5\end{aligned}$$

20 a Let $y = \mathrm{f}(x) = 3x - 2$

Then $\mathrm{f}^{-1}(y) = x = \frac{y+2}{3}$

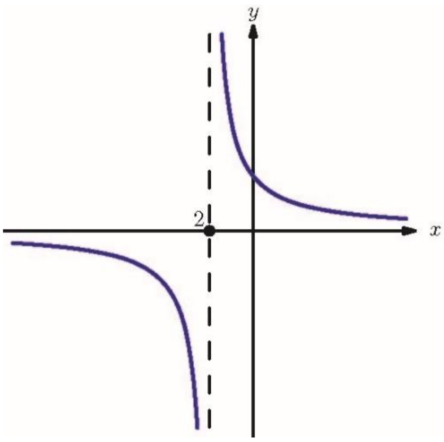
Changing variables: $\mathrm{f}^{-1}(x) = \frac{x+2}{3}$

b

$$(\mathrm{g} \circ \mathrm{f}^{-1})(x) = \mathrm{g}(\mathrm{f}^{-1}(x))$$
$$= \frac{5}{3\mathrm{f}^{-1}(x)}$$
$$= \frac{5}{x+2}$$

c i When $x = 0$, $\mathrm{h}(x) = \frac{5}{2}$

ii



d i When $\mathrm{h}^{-1}(x) = 0, x = \mathrm{h}(0) = \frac{5}{2}$

ii The horizontal asymptote of $y = \mathrm{h}(x)$ is $y = 0$ so the vertical asymptote of $y = \mathrm{h}^{-1}(x)$ is $x = 0$

e $\mathrm{h}^{-1}(a) = 3$ so $a = \mathrm{h}(3) = \frac{5}{3+2} = 1$

21

$$(\mathrm{f} \circ \mathrm{g})(x) = \mathrm{e}^{2(\ln(x-2))}$$
$$= \mathrm{e}^{\ln((x-2)^2)}$$
$$= (x-2)^2$$

Let $y = (\mathrm{f} \circ \mathrm{g})(x) = (x-2)^2$

Then $x = (\mathrm{f} \circ \mathrm{g})^{-1}(y) = 2 + \sqrt{y}$

Changing variables: $(\mathrm{f} \circ \mathrm{g})^{-1}(x) = 2 + \sqrt{x}$

$\mathrm{g}^{-1}(x) = 2 + \mathrm{e}^{x}$ and $\mathrm{f}^{-1}(x) = \frac{1}{2}\ln x = \ln(\sqrt{x})$

$\mathrm{g}^{-1}(\mathrm{f}^{-1}(x)) = 2 + \mathrm{e}^{\ln\sqrt{x}} = 2 + \sqrt{x}$

So, $(\mathrm{f} \circ \mathrm{g})^{-1}(x) \equiv \mathrm{g}^{-1}(\mathrm{f}^{-1}(x))$

22 $f^{-1}(g^{-1}(x)) = 9$ so

$$x = g(f(9))$$
$$= \frac{1}{1+\sqrt{9}} + 7$$
$$= \frac{29}{4}$$

23 **a** $f(x) = 2 + x - x^3$ has local maximum at $x = 0.577$ (GDC) so $f(x)$ is one-to-one for $x \geq 0.577$

b The graph of $y = f(x)$ is the reflection of the graph of $y = f^{-1}(x)$ in the line $y = x$.

c By (b), the intersection of $y = f(x)$ and $y = f^{-1}(x)$ lies on $y = x$ so $f(x) = x$

Substituting: $2 + x - x^3 = x$

$x = \sqrt[3]{2}$

24 **a** **i**

$$(g \circ f)(x) = g(f(x))$$
$$= \frac{1}{2x+3}$$

Domain of $g(x)$ is $x \neq 0$ so require $f(x) \neq 0$: $x \neq -\frac{3}{2}$

ii

$$(f \circ g)(x) = f(g(x))$$
$$= \frac{2}{x} + 3$$

Domain of $f(x)$ is $\mathbb{R}$ so the only restriction is the domain of $g(x)$: $x \neq 0$.

b If $f(x) = (g^{-1} \circ f \circ g)(x)$ then $(g \circ f)(x) = (f \circ g)(x)$

$$\frac{1}{2x+3} = \frac{2}{x} + 3 = \frac{2+3x}{x}$$

Rearranging: $x = (2x+3)(2+3x)$

$6x^2 + 12x + 6 = 0$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1.$

Substituting into $y = f(x)$ gives point of intersection $(-1,1)$

25 $y = x^3 - 2x$

Translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$: Replace x with $(x - 3)$

$$y_1 = (x-3)^3 - 2(x-3)$$
$$= x^3 - 9x^2 + 25x - 21$$

Vertical stretch scale factor 2: $y_2 = 2y_1$

$y_2 = 2x^3 - 18x^2 + 50x - 42$

Worked solutions

26 a $x^2 + 4x + 9 = (x+2)^2 + 5$

b $y_1 = x^2$

Replace x with $(x+2)$: Translation $\begin{pmatrix}-2\\0\end{pmatrix}$

$y_2 = (x+2)^2$

$y_3 = y_2 + 5$: Translation $\begin{pmatrix}0\\5\end{pmatrix}$

The two transformations are both translations, equivalent to $\begin{pmatrix}-2\\5\end{pmatrix}$

27 a $y = \frac{1}{x}$

Translation $\begin{pmatrix}-2\\0\end{pmatrix}$: replace x with $(x+2)$

$$y_1 = \frac{1}{x+2}$$

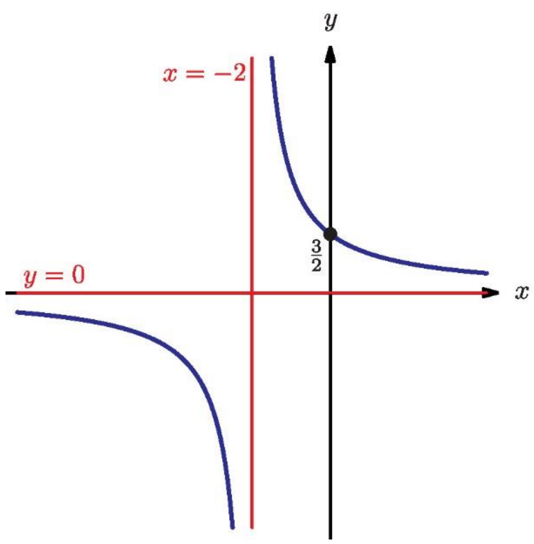
Vertical stretch sf 3: $y_2 = 3y_1$

$$y_2 = \frac{3}{x+2}$$

b $y = \frac{1}{x}$ has asymptotes $y = 0$ and $x = 0$

After the transformations listed in part **a**, the new graph has asymptotes $y = 0$ and $x = -2$.

When $x = 0, y = \frac{3}{2}$ so the graph has y-intercept $\left(0, \frac{3}{2}\right)$.



28 a

$$2 + \frac{1}{x-5} = \frac{2(x-5)+1}{x-5}$$
$$= \frac{2x-10+1}{x-5}$$
$$= \frac{2x-9}{x-5}$$

b $y_1 = \frac{1}{x}$

Replace x with $(x - 5)$: Translation $\begin{pmatrix}5\\0\end{pmatrix}$

$y_2 = \dfrac{1}{x - 5}$

$y_3 = y_2 + 2$: Translation $\begin{pmatrix}0\\2\end{pmatrix}$

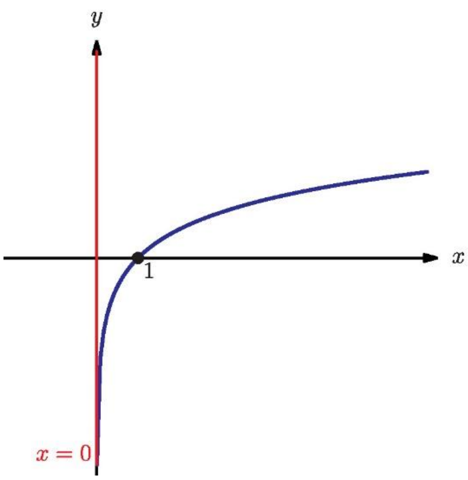
$y_3 = 2 + \dfrac{1}{x - 5}$

The two transformations are both translations, equivalent to $\begin{pmatrix}5\\2\end{pmatrix}$

c $y = \frac{1}{x}$ has asymptotes $x = 0$ and $y = 0$

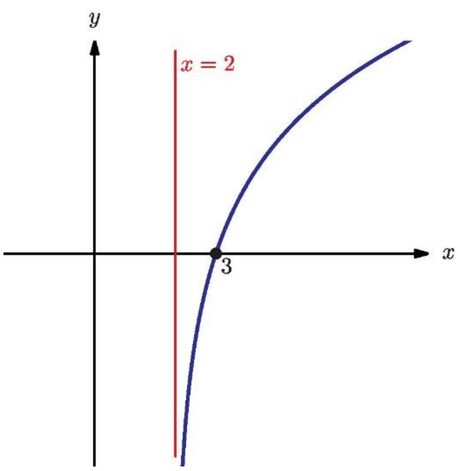
After the translations described in part **b**, the asymptotes to the new graph are $x = 5$ and $y = 2$.

29 a $y = \ln x$ has asymptote $x = 0$ and x-intercept at $(1, 0)$.



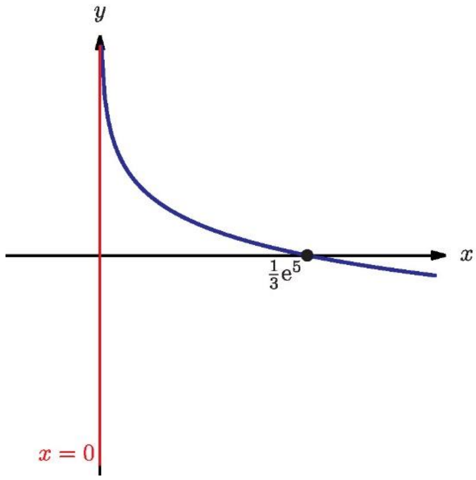
b The graph of $y = 3\ln(x - 2)$ is the graph of $y = \ln x$ after a translation $\begin{pmatrix}2\\0\end{pmatrix}$ and a vertical stretch with scale factor 3.

It has asymptote $x = 2$ and x-intercept $(3, 0)$.



c The graph of $y = 5 - \ln(3x)$ is the graph of $y = \ln x$ after a reflection through the y-axis, a translation $\begin{pmatrix}0\\5\end{pmatrix}$ and a horizontal stretch with scale factor $\frac{1}{3}$.

It has asymptote $x = 0$ and x-intercept $\left(\frac{e^5}{3}, 0\right)$.



30 $y_1 = ax + b$

Translation $\begin{pmatrix}3\\0\end{pmatrix}$: Replace x with $(x - 3)$

$$\begin{aligned} y_2 &= a(x-3) + b \\ &= ax - 3a + b \end{aligned}$$

Vertical stretch with scale factor 7: $y_3 = 7y_2$

$y_3 = 7ax - 21a + 7b$

Reflection in the x-axis: $y_4 = -y_3$

$$\begin{aligned} y_4 &= -7ax + 21a - 7b \\ &= 35 - 21x \end{aligned}$$

Comparing coefficients:

$x^1: -7a = -21 \Rightarrow a = 3$

$x^0: 21a - 7b = 35 \Rightarrow b = 4$

31 $y_1 = 9(x-3)^2$

Replace x with $(x + 5)$: Translation $\begin{pmatrix}-5\\0\end{pmatrix}$

$y_2 = 9(x+2)^2$

$y_3 = \frac{1}{3}y_2$: Vertical stretch with scale factor $\frac{1}{3}$

$y_3 = 3(x+2)^2$

The two transformations (in either order) are a horizontal translation of -5 units and a vertical stretch with scale factor $\frac{1}{3}$.

32 $y_1 = \ln x$

Translation $\begin{pmatrix}2\\0\end{pmatrix}$: Replace x with $(x-2)$

$y_2 = \ln(x-2)$

Translation $\begin{pmatrix}0\\3\end{pmatrix}$: $y_3 = y_2 + 3$

$y_3 = 3 + \ln(x-2)$

Vertical stretch with scale factor 2: $y_4 = 2y_3$

$$\begin{aligned} y_4 &= 6 + 2\ln(x-2) \\ &= \ln e^6 + \ln((x-2)^2) \\ &= \ln(e^6(x-2)^2) \end{aligned}$$

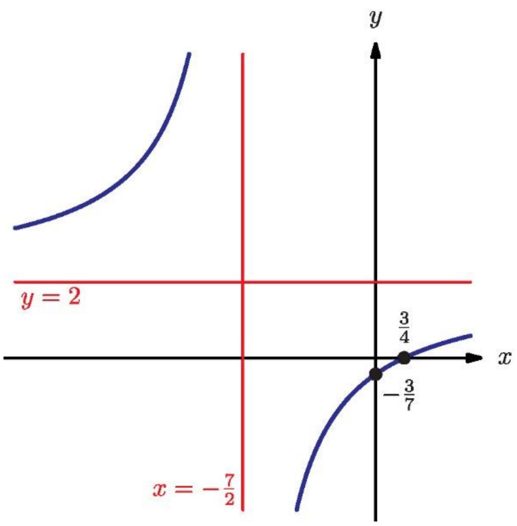
33 a Vertical asymptote is root of denominator: $x = -\frac{7}{2}$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 2$

b When $x = 0, y = -\frac{3}{7}$ so y-intercept is $-\frac{3}{7}$

When $x = \frac{3}{4}, y = 0$ so x-intercept is $\frac{3}{4}$

c



34 a Vertical asymptote is root of denominator: $x = -5$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 3$

b Domain of $\mathrm{f}(x)$ is $x \neq -5$

Range of $\mathrm{f}(x)$ is $f(x) \neq 3$

35 a $y_1 = \ln x$

$y_2 = \ln(x+3)$: Replacing x with $(x+3)$ represents a translation $\begin{pmatrix}-3\\0\end{pmatrix}$

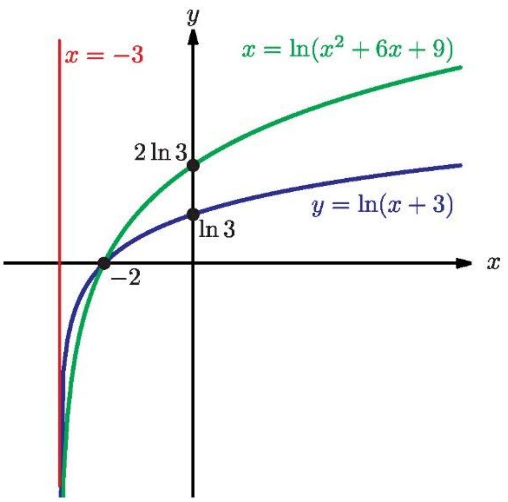
b

$$\log(x^2 + 6x + 9) = \log((x + 3)^2)$$
$$= 2\log|x + 3|$$

The graph of $y = 2\log(x + 3)$ is the graph of $y = \log(x + 3)$, stretched vertically with scale factor 2.

The graph of $y = 2\log|x + 3|$ is the same graph, together with its reflection through $x = -3$.

Since the graph required is only for $x > -3$, the left side of the graph is not needed.



36 a

$$\mathrm{f}(x) = p - (x^2 - qx)$$
$$= p - \left(\left(x - \frac{q}{2}\right)^2 - \frac{q^2}{4}\right)$$
$$= -\left(x - \frac{q}{2}\right)^2 + p + \frac{q^2}{4}$$

The curve has maximum at $\left(\frac{q}{2}, p + \frac{q^2}{4}\right) = (3, 5)$

$$q = 6, p = 5 - \frac{q^2}{4} = -4$$

b $\mathrm{f}(x) = -4 + 6x - x^2$

Translation $\begin{pmatrix}3\\0\end{pmatrix}$: Replace x with $(x - 3)$

$$\mathrm{g}(x) = -4 + 6(x - 3) - (x - 3)^2$$
$$= -4 - 18 + 6x - x^2 + 6x - 9$$
$$= -x^2 + 12x - 31$$

37 a $\mathrm{f}(x) = p + \frac{9}{x-q}$ for $x \neq q$

Vertical asymptote is root of denominator: $x = q = 3$

b $f(0) = p + \frac{9}{-q} = 4$

$p = 4 + \dfrac{9}{q} = 7$

c Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 7$

38 a $N(0) = 840 = 40 + a$

$a = 800$

b $N(4) = 90 = 800 \times b^{-4} + 40$

$b^{-4} = \dfrac{50}{800} = \dfrac{1}{16}$

$b = 2$

c The horizontal asymptote of the graph of $N(t)$ in this model is $N = 40$, below which the curve will not pass.

The minimum number of fish the model predicts is $N = 40$

39 $\mathrm{f}(x) = \ln x$

Translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$: Replace x with $(x - 3)$.

$\mathrm{f}_2(x) = \ln(x - 3)$

Translation $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$: $\mathrm{f}_3(x) = \mathrm{f}_2(x) - 2$

$$\begin{aligned} \mathrm{f}_3(x) &= \ln(x-3) - 2 \\ &= \ln(x-3) - \ln(\mathrm{e}^2) \\ &= \ln\left(\frac{x-3}{\mathrm{e}^2}\right) \end{aligned}$$

Reflection through x-axis: $\mathrm{f}_4(x) = -\mathrm{f}_3(x)$

$$\begin{aligned} \mathrm{f}_4(x) &= -\ln\left(\frac{x-3}{\mathrm{e}^2}\right) \\ &= \ln\left(\frac{\mathrm{e}^2}{x-3}\right) \end{aligned}$$

40 a Vertical asymptote is root of denominator: $x = 5$

Horizontal asymptote is end behaviour as $x \to \pm\infty$: $y = 2$

b

$$\begin{aligned} \mathrm{f}(x) &= \frac{2x - 10 + 1}{x - 5} \\ &= 2 + \frac{1}{x-5} \end{aligned}$$

$\alpha = 2, \beta = 1$

c

$$y = \frac{1}{x}$$

Replace x with $(x-5)$: Translation $\binom{5}{0}$

$$y_2 = \frac{1}{x-5}$$

$y_3 = y_2 + 2$: Translation $\binom{0}{2}$

$$y_3 = 2 + \frac{1}{x-5} = \mathrm{f}(x)$$

d If $y = \mathrm{f}(x) = 2 + \frac{1}{x-5}$

Then $\frac{1}{x-5} = y - 2$

$$x - 5 = \frac{1}{y-2}$$

$$x = 5 + \frac{1}{y-2} = \mathrm{f}^{-1}(y)$$

$$\begin{aligned}\mathrm{f}^{-1}(x) &= 5 + \frac{1}{x-2} \\ &= \frac{5x-9}{x-2}\end{aligned}$$

The inverse has domain equivalent to the range of the original function: $x \neq 2$

e The graph of $\mathrm{f}(x)$ is mapped to the graph of $\mathrm{f}^{-1}(x)$ by a reflection through the line $y = x$.

41

$$\begin{aligned}y &= 4^x \\ &= 2^{2x}\end{aligned}$$

Transforming $y = 2^{2x}$ to $y = 2^x$

Replace x with $\frac{x}{2}$: Horizontal stretch with scale factor 2

42 Using change of base rule:

$$y = \log_{10} x = \frac{\ln x}{\ln 10}$$

Transforming $y = \ln x$ to $y = \left(\frac{1}{\ln 10}\right) \ln x$: Vertical stretch, scale factor $\frac{1}{\ln 10}$

43 The graph of $y = \mathrm{f}(x)$ has roots at $-1, 0$ and 1

The graph of $y = x\mathrm{f}(x)$ will have a single root at -1 and 1, and a double root at 0.

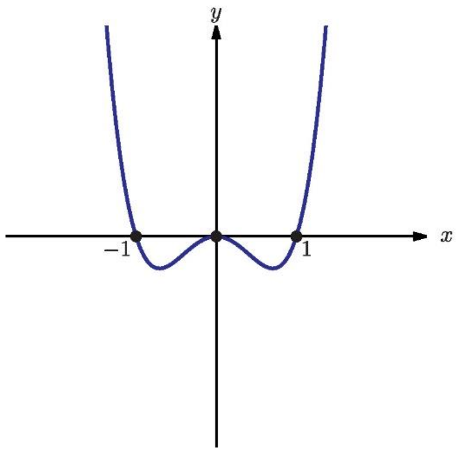
$\mathrm{f}(x) < 0$ for $x < -1$ and $0 < x < 1$

$x\mathrm{f}(x) < 0$ for $-1 < x < 0$ and $0 < x < 1$ (ie opposite sign for negative x, same sign for positive x)

End behaviour:

As $x \to -\infty$, $\mathrm{f}(x) \to -\infty$ so $x\mathrm{f}(x) \to \infty$.

As $x \to \infty$, $\mathrm{f}(x) \to \infty$ so $x\mathrm{f}(x) \to \infty$.



Comment: Strictly speaking, the graph of $f(x)$ shows roots at $-1, 0$ and 1 with odd multiplicity, not necessarily 1, so it would be more accurate to say that $xf(x)$ has roots at ± 1 with odd multiplicity and a root at 0 with even multiplicity, but the end interpretation is equivalent, for the purposes of a sketch.

44 To reflect in the line $y = 1$:

Translate $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$: $y_2 = \mathrm{f}(x) - 1$

Reflect in the x-axis: $y_3 = -y_2 = 1 - \mathrm{f}(x)$

Translate $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $y_4 = 1 + y_3 = 2 - \mathrm{f}(x)$

The new graph has equation $y = 2 - \mathrm{f}(x)$

45 If $\mathrm{f}(x^2) = x^2\mathrm{f}(x)$ for all x in the domain of f then

$\mathrm{f}(x) = \frac{\mathrm{f}(x^2)}{x^2}$ for $x \neq 0$

For any value $a \neq 0$ in the domain of f, $\mathrm{f}(-a) = \frac{\mathrm{f}((-a)^2)}{(-a)^2} = \frac{\mathrm{f}(a^2)}{a^2} = \mathrm{f}(a)$

If $\mathrm{f}(a) = \mathrm{f}(-a)$ for any $a \neq 0$ in the domain of f then the graph of $\mathrm{f}(x)$ is symmetrical across the y-axis.

6 Complex numbers

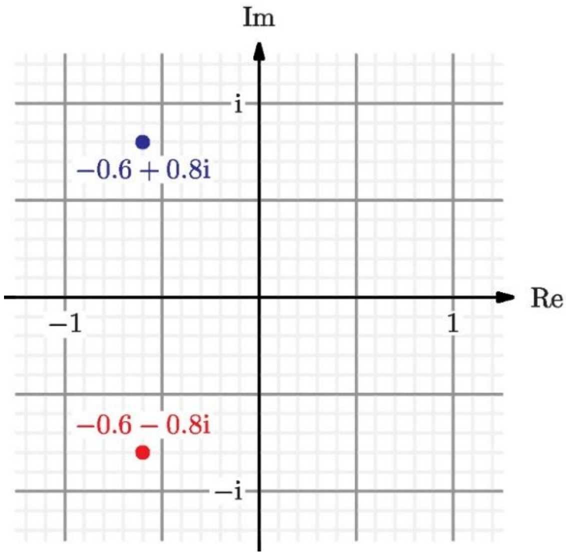
These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 6A

24 a

$$x = \frac{-6 \pm \sqrt{(6^2 - 100)}}{10} = -\frac{3}{5} \pm \frac{4}{5}\text{i}$$

b



25

$$\begin{aligned} z &= 7 + 3\text{i} - \frac{10\text{i}(2-\text{i})}{(2+\text{i})(2-\text{i})} \\ &= 7 + 3\text{i} - \frac{10 + 20\text{i}}{5} \\ &= 5 - \text{i} \end{aligned}$$

So $z^* = 5 + \text{i}$

26

$4z - 23 = 5\text{i}z + 2\text{i}$

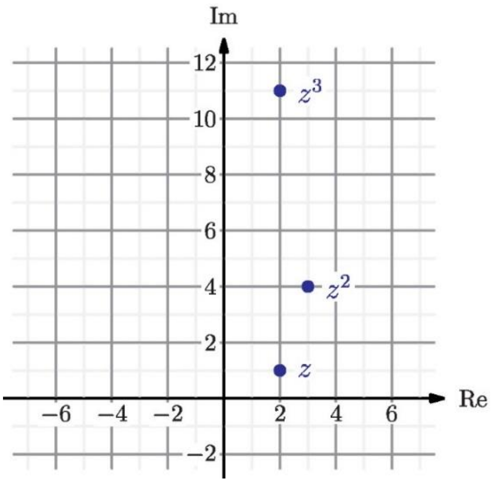
$z(4 - 5\text{i}) = (23 + 2\text{i})$

$$z = \frac{23 + 2\text{i}}{4 - 5\text{i}}$$
$$= \frac{(23 + 2\text{i})(4 + 5\text{i})}{(4 - 5\text{i})(4 + 5\text{i})}$$
$$= \frac{82 + 123\text{i}}{41}$$
$$= 2 + 3\text{i}$$

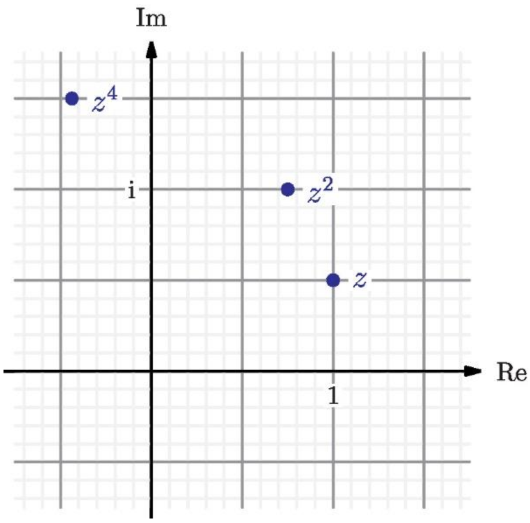
27 a $z^2 = 3 + 4\text{i}$

$z^3 = 2 + 11\text{i}$

b



28



29 a $z + 2w = a + (2 - 2a)\text{i}$

b $zw = 2a - a^2\text{i}$

Worked solutions

30

$$zw = 12 - 3\text{i} = (a - 3\text{i})(1 + b\text{i})$$
$$= a + 3b + (ab - 3)\text{i}$$

$a + 3b = 12$ (1)

$ab - 3 = -3$ (2)

(1): $a = 12 - 3b$

Substituting into (2):

$b(12 - 3b) = 0$

$b = 0$ or 4

So $a = 12, b = 0$ or $a = 0, b = 4$

31 Let $z = x + \text{i}y$ for real values x, y

Then $z^* = x - \text{i}y$

$$3\text{i}z - 2z^* = \text{i} - 4 = (-3y + 3\text{i}x) - (2x - 2\text{i}y)$$
$$= -3y - 2x + (3x + 2y)\text{i}$$

$2x + 3y = 4$ (1)

$3x + 2y = 1$ (2)

$3(2) - 2(1)$: $5x = -5$

$x = -1, y = 2$

$z = -1 + 2\text{i}$

32 Let $z = x + \text{i}y$ for real values x, y

Then $z^* = x - \text{i}y$

$$2z + \text{i}z^* = 11 - 7\text{i} = (2x + 2\text{i}y) + (-y + \text{i}x)$$
$$= 2x - y + (x + 2y)\text{i}$$

$2x - y = 11$ (1)

$x + 2y = -7$ (2)

$(2) + 2(1)$: $5x = 15$

$x = 3, y = -5$

$z = 3 - 5\text{i}$

33 Let $z = x + \text{i}y$ for real values x, y

Then $zz^* = x^2 + y^2$ and $z^2 = x^2 - y^2 + 2xy\text{i}$

$$zz^* - z^2 = 18 + 12\text{i} = x^2 + y^2 - (x^2 - y^2 + 2xy\text{i})$$
$$= 2y^2 - 2xy\text{i}$$

$2y^2 = 18$ (1)

$2xy = -12$ (2)

(1): $y = \pm 3$

So $z = \pm(2 - 3\text{i})$

34

$$\frac{k+\mathrm{i}}{k-\mathrm{i}} = \frac{(k+\mathrm{i})^2}{(k-\mathrm{i})(k+\mathrm{i})}$$
$$= \frac{k^2-1+2k\mathrm{i}}{k^2+1}$$
$$= \frac{k^2-1}{k^2+1} + \frac{2k}{k^2+1}\mathrm{i}$$

35

$$z = \frac{a+3\mathrm{i}}{a-3\mathrm{i}} = \frac{(a+3\mathrm{i})^2}{a^2+9}$$
$$= \frac{a^2-9}{a^2+9} + \frac{6a}{a^2+9}\mathrm{i}$$

If $\mathrm{Re}(z) = 0$ then $a = \pm 3$

36 Let $z = x + \mathrm{i}y$ for real values x, y

Then $z^* = x - \mathrm{i}y$

$z^2 = (x+\mathrm{i}y)^2 = x^2 - y^2 + 2xy\mathrm{i}$

$(z^*)^2 = (x-\mathrm{i}y)^2 = x^2 - y^2 - 2xy\mathrm{i}$

So, $\mathrm{Re}(z^2) = x^2 - y^2 = \mathrm{Re}((z^*)^2)$

And $\mathrm{Im}(z^2) = 2xy = -\mathrm{Im}((z^*)^2)$

Therefore, by definition of conjugate, $(z^2)^* = (z^*)^2$

37

$$\begin{cases} 3z + \mathrm{i}w = 5 - 11\mathrm{i}(1) \\ 2\mathrm{i}z - 3w = -2 + \mathrm{i}(2) \end{cases}$$

$3(1) + \mathrm{i}(2)\text{: } (9-2)z = (15 - 33\mathrm{i}) + (-2\mathrm{i} - 1)$

$7z = 14 - 35\mathrm{i}$

$z = 2 - 5\mathrm{i}$

Substituting into (2): $(4\mathrm{i} + 10) - 3w = -2 + \mathrm{i}$

$3w = 12 + 3\mathrm{i}$

$w = 4 + \mathrm{i}$

Solution: $w = 4 + \mathrm{i}, z = 2 - 5\mathrm{i}$

38

$$\begin{cases} 2z - 3\mathrm{i}w = 9 + \mathrm{i}(1) \\ (1+\mathrm{i})z + 4w = 1 + 10\mathrm{i}(2) \end{cases}$$

$4(1) + 3\mathrm{i}(2)\text{: } (8 + 3\mathrm{i} - 3)z = (36 + 4\mathrm{i}) + (3\mathrm{i} - 30)$

$(5 + 3\mathrm{i})z = 6 + 7\mathrm{i}$

$$z = \frac{6+7\mathrm{i}}{5+3\mathrm{i}} = \frac{(6+7\mathrm{i})(5-3\mathrm{i})}{34} = \frac{51+17\mathrm{i}}{34} = \frac{3}{2} + \frac{1}{2}\mathrm{i}$$

Substituting into (1):

$(3+\mathrm{i}) - 3\mathrm{i}w = 9 + \mathrm{i}$

$3\mathrm{i}w = -6$

$w = 2\mathrm{i}$

Solution: $w = 2\mathrm{i}, z = \frac{3}{2} + \frac{1}{2}\mathrm{i}$

39

$$\begin{aligned} b - a + 9\mathrm{i} &= (1 + a\mathrm{i})(1 + b\mathrm{i}) \\ &= 1 - ab + (a+b)\mathrm{i} \end{aligned}$$

$b - a = 1 - ab \qquad (1)$

$9 = a + b \qquad (2)$

Substituting $b = 9 - a$ into (1):

$9 - 2a = 1 - a(9 - a)$

$a^2 - 7a - 8 = 0$

$(a-8)(a+1) = 0$

$a = 8$ or $= -1$

Solutions: $a = 8, b = 1$ or $a = -1, b = 10$

40

$$\begin{aligned} z &= \frac{7+\mathrm{i}}{2-\mathrm{i}} - \frac{3+\mathrm{i}}{a+2\mathrm{i}} \\ &= \frac{(7+\mathrm{i})(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})} - \frac{(3+\mathrm{i})(a-2\mathrm{i})}{(a+2\mathrm{i})(a-2\mathrm{i})} \\ &= \frac{13+9\mathrm{i}}{5} - \frac{3a+2+(a-6)\mathrm{i}}{a^2+4} \\ &= \frac{13(a^2+4) - 15a - 10}{5(a^2+4)} + \frac{9(a^2+4) - 5a + 30}{5(a^2+4)}\mathrm{i} \end{aligned}$$

If $\mathrm{Re}(z) = \mathrm{Im}(z)$, then

$13(a^2+4) - 15a - 10 = 9(a^2+4) - 5a + 30$

$4a^2 - 10a - 24 = 0$

$2a^2 - 5a - 12 = 0$

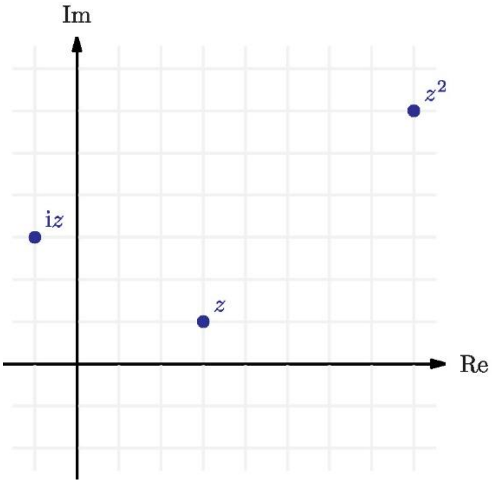
$(a-4)(2a+3) = 0$

$a = 4$ or $-\frac{3}{2}$

Worked solutions

Exercise 6B

16



17 a $|-2+2\text{i}| = \sqrt{(-2)^2+2^2} = \sqrt{8}$

b $\text{Im } z > 0$ so $\arg z = \arctan\left(\frac{2}{-2}\right) = \frac{3\pi}{4}$

c $|z^2| = |z|^2 = 8$

$$\arg z^2 = 2\arg \text{z} = \frac{3\pi}{2}$$

d $z^2 = 8\left(\cos\left(\frac{3\pi}{2}\right) + \text{i}\sin\left(\frac{3\pi}{2}\right)\right) = -8\text{i}$

18 $\text{cis}\, 0.6 \times \text{cis}\, 0.4 = \text{cis}\, 1$

19 a $\text{Im }(w) > 0$ so $\arg w = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

b $\text{Im }(z) > 0$ so $\arg z = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$$\text{Im}(zw) = \text{Im}(z) + \text{Im}(w) = \frac{7\pi}{12}$$

20 a $\text{cis}\frac{\pi}{3} \times \text{cis}\frac{\pi}{6} = \text{cis}\frac{\pi}{2} = \text{i}$

b $\text{cis}\frac{\pi}{3} + \text{cis}\frac{\pi}{6} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}\right) + \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}\right) = \frac{1+\sqrt{3}}{2}(1+\text{i})$

21 a $|2-2\text{i}| = \sqrt{2^2+(-2)^2} = 2\sqrt{2}$

$$\text{Re }(2-2\text{i}) > 0 \text{ so } \arg(2-2\text{i}) = \arctan\left(-\frac{2}{2}\right) = -\frac{\pi}{4}$$

$$z = 2-2\text{i} = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

b $z^5 = \left(2\sqrt{2}\right)^5 \text{cis}\left(-\frac{5\pi}{4}\right)$

$= 128\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\text{i}\right)$

$= -128 + 128\text{i}$

22 a $\left|\sqrt{3}+\mathrm{i}\right| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$

$\mathrm{Re}\left(\sqrt{3}+\mathrm{i}\right) > 0$ so $\arg\left(\sqrt{3}+\mathrm{i}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$z = \sqrt{3}+\mathrm{i} = 2\,\mathrm{cis}\left(\frac{\pi}{6}\right)$

b $z^3 = (2)^3\,\mathrm{cis}\left(\frac{3\pi}{6}\right)$

$= 8\mathrm{i}$

23 a $\left|-\sqrt{2}-\mathrm{i}\sqrt{2}\right| = \sqrt{\left(-\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = 2$

$\mathrm{Re}\left(-\sqrt{2}-\mathrm{i}\sqrt{2}\right) < 0$ so $\arg(2-2\mathrm{i}) = \pi + \arctan\left(\frac{-\sqrt{2}}{-\sqrt{2}}\right) = \frac{5\pi}{4}$

$w = -\sqrt{2}-\mathrm{i}\sqrt{2} = 2\,\mathrm{cis}\left(\frac{5\pi}{4}\right)$

b $w^6 = 2^6\,\mathrm{cis}\left(\frac{30\pi}{4}\right) = 64\,\mathrm{cis}\left(-\frac{\pi}{2}\right) = -64\mathrm{i}$

$z = \mathrm{cis}\left(-\frac{\pi}{7}\right)$ so $z^7 = \mathrm{cis}(-\pi) = -1$

Then $w^6z^7 = 64\mathrm{i}$

24 $|3-3\mathrm{i}| = \sqrt{3^2+(-3)^2} = 3\sqrt{2}$

$\mathrm{Re}\,(3-3\mathrm{i}) > 0$ so $\arg(3-3\mathrm{i}) = \arctan\left(\frac{-3}{3}\right) = -\frac{\pi}{4}$

So $w = 3-3\mathrm{i} = 3\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$

Then $w^4 = \left(3\sqrt{2}\right)^4\mathrm{cis}(-\pi) = -324$

$z = \mathrm{cis}\left(\frac{3\pi}{8}\right)$ so $z^6 = \mathrm{cis}\left(\frac{9\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}$

Hence $w^4z^6 = -162\sqrt{2} - \left(162\sqrt{2}\right)\mathrm{i}$

25 $z = 2\,\mathrm{cis}\frac{7\pi}{24}$ so $z^n = 2^n\,\mathrm{cis}\frac{7n\pi}{24}$

For this to be a real number, $\frac{7n}{24} \in \mathbb{Z}$

The least value $n \in \mathbb{Z}^+$ under this condition is $n = 24$.

26 $z = \mathrm{cis}\frac{5\pi}{18}$ so $z^n = \mathrm{cis}\frac{5n\pi}{18}$

$\mathrm{i} = \mathrm{cis}\left(\frac{\pi}{2} + 2k\pi\right)$ for any $k \in \mathbb{Z}$

Require $\frac{5n}{18} = \frac{1}{2} + 2k$

$5n = 9 + 36k$

The least such $n \in \mathbb{Z}^+$ is $n = 9$, with $k = 1$.

27 $|zw| = |z||w| = \sqrt{\left(-4\sqrt{2}\right)^2 + \left(4\sqrt{2}\right)^2} = 8$

$|zw^{-1}| = |z||w|^{-1}\sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$

Then $\left|\dfrac{zw}{zw^{-1}}\right| = 4 = |w|^2$ so $|w| = 2$ and $|z| = 4$

$\arg(zw) = \arg z + \arg w = \dfrac{\pi}{2} + \arctan\left(\dfrac{4\sqrt{2}}{-4\sqrt{2}}\right) = \dfrac{3\pi}{4}$

$\arg(zw^{-1}) = \arg z - \arg w = \arctan\left(\dfrac{\sqrt{3}}{1}\right) = \dfrac{\pi}{3}$

Then $\arg\left(\dfrac{zw}{zw^{-1}}\right) = 2\arg w = \dfrac{3\pi}{4} - \dfrac{\pi}{3} = \dfrac{5\pi}{12}$ so $\arg w = \dfrac{5\pi}{24}$ and $\arg z = \dfrac{13\pi}{24}$

28 **a** **i** $|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

ii $\operatorname{Re} z > 0$ so $\arg z = \arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) = \arctan 1 = \frac{\pi}{4}$

iii $|w| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = 2$

iv $\operatorname{Re} w > 0$ so $\arg w = \arctan\left(\frac{\operatorname{Im} w}{\operatorname{Re} w}\right) = \arctan\sqrt{3} = \frac{\pi}{3}$

b

$\left|\dfrac{w}{z}\right| = \dfrac{|w|}{|z|} = 2$

$\arg\left(\dfrac{w}{z}\right) = \arg w - \arg z = \dfrac{\pi}{12}$

So $\dfrac{w}{z} = 2\operatorname{cis}\dfrac{\pi}{12}$

c

$$\begin{aligned}
\frac{w}{z} &= \frac{1 + \mathrm{i}\sqrt{3}}{\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}} \\
&= \sqrt{2}\left(\frac{1 + \mathrm{i}\sqrt{3}}{1 + \mathrm{i}}\right) \\
&= \sqrt{2}\left(\frac{1 + \mathrm{i}\sqrt{3}}{1 + \mathrm{i}} \times \frac{1 - \mathrm{i}}{1 - \mathrm{i}}\right) \\
&= \frac{\sqrt{2}\left(1 + \sqrt{3} + \mathrm{i}\left(\sqrt{3} - 1\right)\right)}{2} \\
&= \frac{1 + \sqrt{3}}{\sqrt{2}} + \mathrm{i}\frac{\sqrt{3} - 1}{\sqrt{2}} \\
&= \frac{\sqrt{2} + \sqrt{6}}{2} + \mathrm{i}\frac{\sqrt{6} - \sqrt{2}}{2}
\end{aligned}$$

d From part **b**, $\frac{w}{z} = 2\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

Equating the real part with the result in part **c**,

$$\cos\frac{\pi}{12} = \frac{1}{2}\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$$

29 a $5 + 2\mathrm{e}^{3\mathrm{i}} = 5 + 2\cos 3 + 2\mathrm{i}\sin 3 \approx 3.02 + 0.282\mathrm{i}$

$|5 + 2\mathrm{e}^{3\mathrm{i}}| \approx \sqrt{3.02^2 + 0.282^2} \approx 3.03$

$\mathrm{Re}\left(5 + 2\mathrm{e}^{3\mathrm{i}}\right) > 0$ so $\arg(5 + 2\mathrm{e}^{3\mathrm{i}}) \approx \arctan\left(\frac{0.282}{3.02}\right) \approx 0.093$

So $5 + 2\mathrm{e}^{3\mathrm{i}} \approx 3.03\mathrm{e}^{0.093\mathrm{i}}$

b Let $z_1 = 5 + 2e^{3\mathrm{i}}$ and $z_2 = e^{\mathrm{i}\theta} = (\cos\theta + \mathrm{i}\sin\theta)$

$z_1 z_2 = 5\mathrm{e}^{\mathrm{i}\theta} + 2\mathrm{e}^{(3+\theta)\mathrm{i}}$

$$\begin{aligned}\text{Then } \mathrm{Im}(z_1 z_2) &= 5\sin\theta + 2\sin(3+\theta) \\ &= \mathrm{Im}\left(3.03\mathrm{e}^{(0.093+\theta)\mathrm{i}}\right) \\ &= 3.03\sin(0.093+\theta)\end{aligned}$$

30 a $3 - 2\mathrm{e}^{10\mathrm{i}} = 3 - 2\cos 10 - 2\mathrm{i}\sin 10 \approx 4.68 + 1.09\mathrm{i}$

$|3 - 2\mathrm{e}^{10\mathrm{i}}| \approx \sqrt{4.68^2 + 1.09^2} \approx 4.80$

$\mathrm{Re}\left(3 - 2\mathrm{e}^{10\mathrm{i}}\right) > 0$ so $\arg(3 - 2\mathrm{e}^{10\mathrm{i}}) \approx \arctan\left(\frac{1.09}{4.68}\right) \approx 0.229$

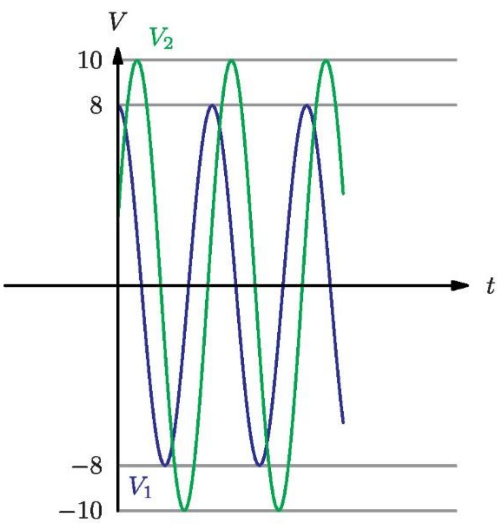
So $3 - 2\mathrm{e}^{10\mathrm{i}} \approx 4.80\mathrm{e}^{0.229\mathrm{i}}$

b Let $z_1 = 3 - 2e^{10\mathrm{i}}$ and $z_2 = e^{\mathrm{i}t} = (\cos t + \mathrm{i}\sin t)$

$z_1 z_2 = 3\mathrm{e}^{\mathrm{i}t} - 2\mathrm{e}^{(10+t)\mathrm{i}}$

$$\begin{aligned}\text{Then } \mathrm{Re}(z_1 z_2) &= 3\cos t - 2\cos(10+t) \\ &= \mathrm{Re}\left(4.80\mathrm{e}^{(0.229+t)\mathrm{i}}\right) \\ &= 4.80\cos(0.229+t)\end{aligned}$$

31 a



b Let $z_1 = 8+10\mathrm{e}^{5\mathrm{i}} = 8 + 10\cos 5 + 10\mathrm{i}\sin 5 \approx 10.8 - 9.59\mathrm{i}$

$|8+10\mathrm{e}^{5\mathrm{i}}| \approx \sqrt{10.8^2 + 9.59^2} \approx 14.5$

$\mathrm{Re}\left(8+10\mathrm{e}^{5\mathrm{i}}\right) > 0$ so $\arg(8+10\mathrm{e}^{5\mathrm{i}}) \approx \arctan\left(-\frac{9.59}{10.8}\right) \approx -0.724$

So $8+10\mathrm{e}^{5\mathrm{i}} \approx 14.5\mathrm{e}^{-0.724\mathrm{i}}$

Let $z_2 = \mathrm{e}^{30\mathrm{i}t} = (\cos 30t + \mathrm{i}\sin 30t)$

$z_1 z_2 = 8\mathrm{e}^{30\mathrm{i}t} + 10\mathrm{e}^{(5+30t)\mathrm{i}}$

$$\begin{aligned}\text{Then } \mathrm{Re}(z_1z_2) &= 8\cos 30t + 10\cos(5+30t)\\ &= \mathrm{Re}\left(14.5\mathrm{e}^{(-0.724+30t)\mathrm{i}}\right)\\ &= 14.5\cos(30t - 0.724)\end{aligned}$$

32 $B = 200 + 40\sin\left(\frac{2\pi}{365}t\right), W = 60 + 20\sin\left(\frac{2\pi}{365}t + 2\right)$

$B + W = 260 + 40\sin\left(\frac{2\pi}{365}t\right) + 20\sin\left(\frac{2\pi}{365}t + 2\right)$

Let $z_1 = 40 + 20e^{2\mathrm{i}} = 40 + 20\cos 2 + 20\mathrm{i}\sin 2 \approx 31.7 + 18.2\mathrm{i}$

$|40 + 20\mathrm{e}^{2\mathrm{i}}| \approx \sqrt{31.7^2 + 18.2^2} \approx 36.5$

$\mathrm{Re}\left(40 + 20\mathrm{e}^{2\mathrm{i}}\right) > 0$ so $\arg(40 + 20\mathrm{e}^{2\mathrm{i}}) \approx \arctan\left(\frac{18.2}{31.7}\right) \approx 0.093$

So $40 + 20\mathrm{e}^{2\mathrm{i}} \approx 36.5\mathrm{e}^{0.521\mathrm{i}}$

Let $z_2 = e^{\frac{2\pi}{365}t\mathrm{i}} = \left(\cos\left(\frac{2\pi}{365}t\right) + \mathrm{i}\sin\left(\frac{2\pi}{365}t\right)\right)$

$z_1 z_2 = 40\mathrm{e}^{\frac{2\pi\mathrm{i}}{365}\mathrm{t}} + 20\mathrm{e}^{\left(\frac{2\pi}{365}t+2\right)\mathrm{i}}$

$$\begin{aligned}\text{Then } \mathrm{Im}(z_1z_2) &= 40\sin\left(\frac{2\pi}{365}t\right) + 20\sin\left(\frac{2\pi}{365}t + 2\right)\\ &= \mathrm{Im}\left(36.5\mathrm{e}^{0.521+\frac{2\pi}{365}t}\right)\\ &= 36.5\sin\left(0.521 + \frac{2\pi}{365}t\right)\end{aligned}$$

So $B + W = 260 + 36.5\sin\left(\frac{2\pi}{365}t + 0.521\right)$

33 a $\mathrm{i} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}}$ so $\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{-\frac{\pi}{2}} \in \mathbb{R}$

b Using the approximation $\mathrm{e} \approx \pi \approx 3$, $\mathrm{i}^{\mathrm{i}} \approx 3^{-1.5} \approx 0.2$

34 a $-2 = 2\mathrm{e}^{\mathrm{i}\pi}$

b $\ln(-2) = \ln\left(2\mathrm{e}^{\mathrm{i}\pi}\right) = \ln 2 + \mathrm{i}\pi$

35 a $\mathrm{i} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}}$

b $\ln \mathrm{i} = \frac{\mathrm{i}\pi}{2}$

c The argument is not unique; more generally, $\mathrm{i} = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2n\pi\right)}$ for $n \in \mathbb{Z}$.

So $\ln i = i\left(\frac{\pi}{2} + 2n\pi\right)$ for any $n \in \mathbb{Z}$.

Exercise 6C

3 **a** $|a| = \sqrt{4^2 + 1^2} = \sqrt{17}$

$\operatorname{Im} a > 0$ so $\arg a = \arctan\left(\frac{1}{4}\right) = 0.245$

$|b| = \sqrt{5^2 + 3^2} = \sqrt{34}$

$\operatorname{Im} b > 0$ so $\arg b = \arctan\left(\frac{3}{5}\right) = 0.540$

b $\left|\frac{b}{a}\right| = \sqrt{2}$

$\operatorname{Im}\left(\frac{b}{a}\right) = \operatorname{Im} b - \operatorname{Im} a = 0.295$

The mapping is an enlargement scale factor $\sqrt{2}$ and a rotation by 0.295 radians.

4 **a** $|p| = \sqrt{3^2 + 5^2} = \sqrt{34}$

$|q| = \sqrt{\left(\sqrt{30}\right)^2 + 2^2} = \sqrt{34} = |p|$

b $\operatorname{Im} p > 0$ so $\arg a = \arctan\left(\frac{5}{3}\right) = 1.030$

$\operatorname{Im} q < 0$ so $\arg a = \pi + \arctan\left(\frac{2}{-\sqrt{30}}\right) = 2.79$

So P is mapped to Q by a rotation of 1.76 radians clockwise.

5 B is produced from a rotation of point A by 30° anticlockwise about the origin.

This is equivalent to multiplication by $e^{\frac{i\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$$B = (3 + 2i)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= \frac{3\sqrt{3} - 2}{2} + \left(\frac{3 + 2\sqrt{3}}{2}\right)i$$

6 C is produced by rotating A 90° clockwise about the origin.

This is equivalent to multiplication of the complex number represented by A by $e^{-\frac{i\pi}{2}} = -i$

$c = (5 + 2i)(-i) = 2 - 5i$ so C has coordinates $(2, -5)$

$b = a + c = 7 - 3i$ so B has coordinates $(7, -3)$

7 **a** If a and b are the complex numbers represented by A and B, respectively.

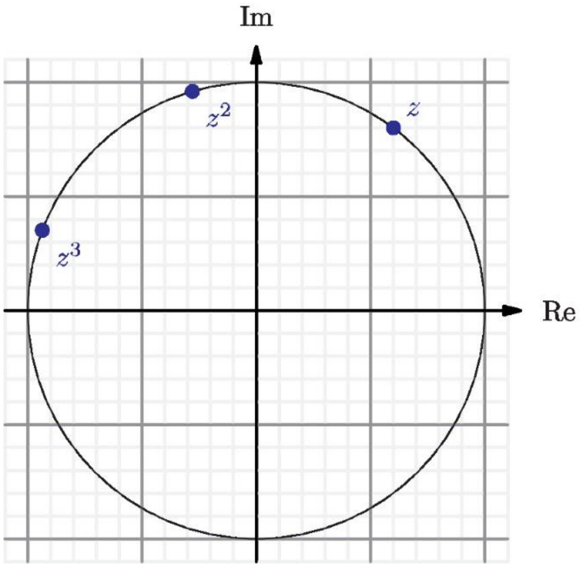
$a = 6 + 3i$ so $|a| = \sqrt{6^2 + 3^2} = 3\sqrt{5}$

$|OB| = |b| = \frac{|a|}{\cos 30°} = 2\sqrt{15}$

b B is produced from A by rotating by $\frac{\pi}{6}$ anticlockwise, equivalent to multiplying by $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{I}\right)$, and then stretching with a scale factor $\sec 30°$.

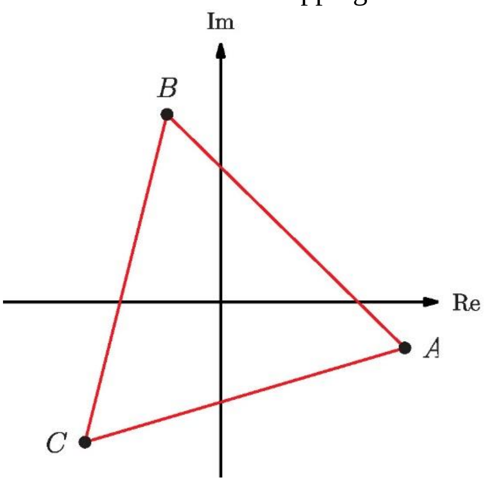
$$\begin{aligned} b &= \frac{2}{\sqrt{3}}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)(6+3\mathrm{i}) \\ &= \left(\frac{\sqrt{3}}{3}+\mathrm{i}\right)(6+3\mathrm{i}) \\ &= 2\sqrt{3}-3+\left(\sqrt{3}+6\right)\mathrm{i} \end{aligned}$$

8 **a** $|z| = \sqrt{0.6^2+0.8^2} = 1$ so z and all powers of z will lie on the unit circle.



b $\arg z = \arctan\left(\frac{0.8}{0.6}\right) = 0.927$

So the transformation mapping z to z^3 is a rotation by $2 \times 0.927 = 1.85$.



9 **a** $a = 4 - \mathrm{i}$

b Each of the other coordinates is achieved by a rotation in either direction of $\frac{2\pi}{3}$, equivalent to multiplication by $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$.

$$b = (4 - \mathrm{i})\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)$$

$$= -2 + \frac{\sqrt{3}}{2} + \left(2\sqrt{3} + \frac{1}{2}\right)\mathrm{i}$$

$$c = (4 - \mathrm{i})\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)$$

$$= -2 - \frac{\sqrt{3}}{2} + \left(-2\sqrt{3} + \frac{1}{2}\right)\mathrm{i}$$

B has coordinates $\left(-2 + \frac{\sqrt{3}}{2}, 2\sqrt{3} + \frac{1}{2}\right)$

C has coordinates $\left(-2 - \frac{\sqrt{3}}{2}, -2\sqrt{3} + \frac{1}{2}\right)$

10 a A rotation θ about point A representing a can be implemented as:

- Translation taking a to the origin: equivalent to subtracting a
- Rotation about the origin by θ: equivalent to multiplying by $\mathrm{e}^{\mathrm{i}\theta}$
- Translation taking the origin back to a.: equivalent to adding a

So $q = (p - a)\mathrm{e}^{\mathrm{i}\theta} + a$

Then

$q - a = (p - a)\mathrm{e}^{\mathrm{i}\theta}$

b

$$p = 1 + 3\mathrm{i}, a = 2 - \mathrm{i}, \theta = \frac{\pi}{3} \text{ so } \mathrm{e}^{\mathrm{i}\theta} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

$$q = \left(1 + 3\mathrm{i} - (2 - \mathrm{i})\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) + (2 - \mathrm{i})$$

$$= (-1 + 4\mathrm{i})\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) + 2 - \mathrm{i}$$

$$= -\frac{1}{2} - 2\sqrt{3} + \left(2 - \frac{\sqrt{3}}{2}\right)\mathrm{i} + 2 - \mathrm{i}$$

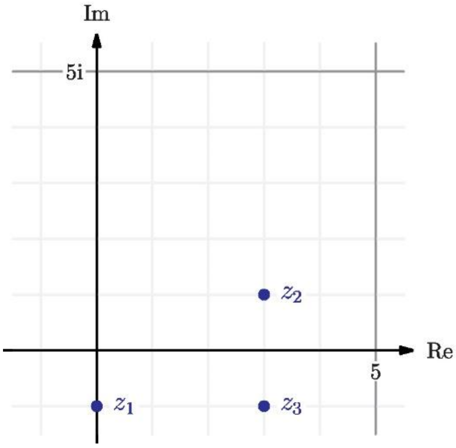
$$= \frac{3}{2} - 2\sqrt{3} + \left(1 - \frac{\sqrt{3}}{2}\right)\mathrm{i}$$

Mixed Practice 6

1 a $z_1 = \frac{1}{\mathrm{i}} = -\mathrm{i}$

b $z_2 = (1 + \mathrm{i})(2 - \mathrm{i}) = 3 + \mathrm{i}$

c $z_3 = z_2^* = 3 - \mathrm{i}$

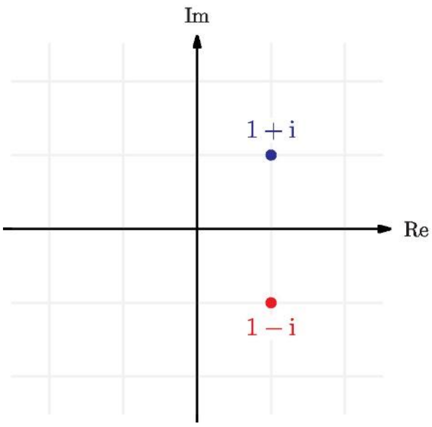


2 **a**

$$\begin{aligned} x^2 - 2x + 2 &= (x-1)^2 + 1 \\ &= (x - 1 + \mathrm{i})(x - 1 - \mathrm{i}) \end{aligned}$$

$x = 1 \pm \mathrm{i}$

b

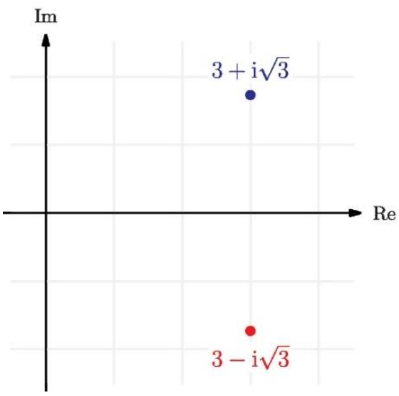


3 **a**

$$\begin{aligned} x^2 - 6x + 12 &= (x-3)^2 + 3 \\ &= \left(x - 3 + \mathrm{i}\sqrt{3}\right)\left(x - 3 - \mathrm{i}\sqrt{3}\right) \end{aligned}$$

$$x = 3 \pm \mathrm{i}\sqrt{3}$$

b



4

$$\begin{aligned} z &= \frac{1+\mathrm{i}}{1+2\mathrm{i}} \\ &= \frac{1+\mathrm{i}}{1+2\mathrm{i}} \times \frac{1-2\mathrm{i}}{1-2\mathrm{i}} \\ &= \frac{3-\mathrm{i}}{5} \\ &= \frac{3}{5} - \frac{1}{5}\mathrm{i} \end{aligned}$$

So $z^* = \frac{3}{5} + \frac{1}{5}\mathrm{i}$

5

$$\frac{z}{z+\mathrm{i}} = 1 + 2\mathrm{i}$$

$z = (1+2\mathrm{i})(z+\mathrm{i})$

$z = z + 2\mathrm{i}z - 2 + \mathrm{i}$

$2\mathrm{i}z = 2 - \mathrm{i}$

$$z = -\frac{1}{2} - \mathrm{i}$$

6 $z + \mathrm{i} = 2z^*$

Let $z = x + \mathrm{i}y$ where $x = \mathrm{Re}\,(z), y = \mathrm{Im}\,(z)$ so $z^* = x - \mathrm{i}y$

$x + (y+1)\mathrm{i} = 2x - 2\mathrm{i}y$

Comparing real and imaginary parts: $x = 2x$ so $x = 0$

$$y + 1 = -2y \text{ so } y = -\frac{1}{3}$$

$$z = -\frac{1}{3}\mathrm{i}$$

7 $z + 4 = \mathrm{i}z$

$z(\mathrm{i} - 1) = 4$

$$z = \frac{4}{\mathrm{i} - 1}$$

$z = -2 - 2\mathrm{i}$

8 **a** $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Since $\mathrm{Re}\, z > 0$, $\arg z = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

b

$$z = \sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{4}\right) \text{ so } z^6 = 8\,\mathrm{cis}\left(\frac{3\pi}{2}\right) = -8\mathrm{i}$$

$$w = \mathrm{cis}\left(\frac{3\pi}{5}\right) \text{ so } w^5 = \mathrm{cis}(3\pi) = -1$$

$$\text{Then } z^6w^5 = 8\,\mathrm{cis}\left(\frac{3\pi}{2} + \pi\right) = 8\,\mathrm{cis}\left(\frac{5\pi}{2}\right) = 8\mathrm{i}$$

9 $|z| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

Since $\text{Re}\, z > 0$, $\arg z = \arctan\left(\frac{-2}{2}\right) = -\frac{\pi}{4}$

Then

$|(z^*)^3| = \left(2\sqrt{2}\right)^3 = 16\sqrt{2}$

$\arg((z^*)^3) = \frac{3\pi}{4}$

10 a Comparing real and imaginary parts:

$p = 3$

$4q = 2$ so $q = \frac{1}{2}$

b If $p = a + \mathrm{i}b$ and $q = p^* = a - \mathrm{i}b$

$a\mathrm{i} - b + 4a - 4\mathrm{i}b = 2 + 3\mathrm{i}$

Comparing real and imaginary parts:

$4a - b = 2(1)$

$a - 4b = 3(2)$

$4(1) - (2): 15a = 5$ so $a = \frac{1}{3}$

$(1): b = 4a - 2 = -\frac{2}{3}$

$p = \frac{1}{3} - \frac{2}{3}\mathrm{i}, q = \frac{1}{3} + \frac{2}{3}\mathrm{i}$

11 a $\text{cis}\left(\frac{\pi}{2}\right) \div \text{cis}\left(\frac{\pi}{6}\right) = \text{cis}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \text{cis}\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

b $\text{cis}\left(\frac{\pi}{2}\right) - \text{cis}\left(\frac{\pi}{6}\right) = \mathrm{i} - \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

12

$$\frac{1}{a + \mathrm{i}} = \frac{1}{a + \mathrm{i}} \times \frac{a - \mathrm{i}}{a - \mathrm{i}} = \frac{a}{a^2 + 1} - \frac{1}{a^2 + 1}\mathrm{i}$$

13

$$\frac{1}{z + \mathrm{i}} = -\frac{2}{z - \mathrm{i}}$$

Taking reciprocals on both sides:

$z + \mathrm{i} = -\frac{z - \mathrm{i}}{2}$

$\frac{3}{2}z = -\frac{1}{2}\mathrm{i}$

$z = -\frac{1}{3}\mathrm{i}$

Worked solutions

14 Let $z = x + \mathrm{i}y$ for real x, y

Then $z^* = x - \mathrm{i}y$

$z + z^* = 2x = 8$ so $x = 4$

$z - z^* = 2\mathrm{i}y = 6\mathrm{i}$ so $y = 3$

$z = 4 + 3\mathrm{i}$

15 $|zw| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2,$

Since $\mathrm{Re}(zw) < 0, \arg(zw) = \pi + \arctan\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$

$\left|\frac{z}{w}\right| = \sqrt{\left(-\frac{1}{2}\right)^2} = \frac{1}{2},$

$\arg\left(\frac{z}{w}\right) = -\frac{\pi}{2}$

$|zw| \times \left|\frac{z}{w}\right| = |z^2| = 1$ so $|z| = 1$

$\arg(zw) + \arg\left(\frac{z}{w}\right) = \arg(z^2) = \frac{\pi}{3}$ so $\arg z = \frac{\pi}{6}$

16

$$\frac{2}{2+\mathrm{i}} - \frac{1}{b+\mathrm{i}} = \frac{2(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})} - \frac{b-\mathrm{i}}{(b+\mathrm{i})(b-\mathrm{i})}$$
$$= \frac{4-2\mathrm{i}}{5} - \frac{b-\mathrm{i}}{b^2+1}$$

If this is a real value then its imaginary part equals zero.

$-\frac{2}{5} + \frac{1}{b^2+1} = 0$

$2(b^2 + 1) = 5$

$b^2 + 1 = \frac{5}{2}$

$b = \pm\sqrt{\frac{3}{2}}$

17 Let $z = 2 + \mathrm{i}y$ for $y \in \mathbb{R}$

Then $z^2 = 4 - y^2 + 4\mathrm{i}y$

$4 - y^2 = 3$ so $y = \pm 1$

$z = 2 \pm \mathrm{i}$

18 Let $z = x + \mathrm{i}y$ for $x, y \in \mathbb{R}$

$z + |z| = 18 + 12\mathrm{i}$

Comparing real and imaginary parts:

$x + \sqrt{x^2 + y^2} = 18$

$y = 12$

$x^2 + 144 = (18 - x)^2 = x^2 + 324 - 36x$

$36x = 180$

$x = 5$

$z = 5 + 12\mathrm{i}$

19 For a polynomial with real coefficients, complex roots occur in conjugate pairs

So, given $1 + 2\mathrm{i}$ is a root, $1 - 2\mathrm{i}$ is also a root.

$(x - 1 - 2\mathrm{i})(x - 1 + 2\mathrm{i}) = 0$

$x^2 - 2x + 5 = 0$

$b = -2, c = 5$

20 Let $z = x + \mathrm{i}y$ for real x, y

Then $|z| = \sqrt{x^2 + y^2}$

$\sqrt{x^2 + y^2} + x + \mathrm{i}y = 8 + 4\mathrm{i}$

Imaginary parts: $y = 4$

Real parts: $\sqrt{x^2 + 16} + x = 8$

$x^2 + 16 = (8 - x)^2 = 64 - 16x + x^2$

$16x = 48$

$x = 3$

$z = 3 + 4\mathrm{i}$

21 a $0.3 + 0.4\mathrm{e}^{2\mathrm{i}} = 0.3 + 0.4\cos 2 + (0.4\sin 2)\mathrm{i} \approx 0.134 + 0.364\mathrm{i}$

$|0.3 + 0.4\mathrm{e}^{2\mathrm{i}}| = \sqrt{0.134^2 + 0.364^2} \approx 0.387$

$\mathrm{Re}\left(0.3 + 0.4\mathrm{e}^{2\mathrm{i}}\right) > 0$ so $\arg(0.3 + 0.4\mathrm{e}^{2\mathrm{i}}) \approx \arctan\left(\frac{0.364}{0.134}\right) \approx 1.22$

So $0.3 + 0.4\mathrm{e}^{2\mathrm{i}} \approx 0.387\mathrm{e}^{1.22\mathrm{i}}$

b Let $z_1 = 0.3 + 0.4\mathrm{e}^{2\mathrm{i}}$ and $z_2 = e^{3\mathrm{i}t} = (\cos 3t + \mathrm{i}\sin 3t)$

$z_1 z_2 = 0.3\mathrm{e}^{3\mathrm{i}t} + 0.4\mathrm{e}^{(2+3t)\mathrm{i}}$

Then $\mathrm{Re}(z_1 z_2) = 0.3\cos 3t + 0.4\cos(2 + 3t)$

$$= \mathrm{Re}\left(0.387\mathrm{e}^{(1.22+3t)\mathrm{i}}\right)$$

$$= 0.387\cos(1.22 + 3t)$$

22 a $|-4+\mathrm{i}| = \sqrt{4^2+1^2} = \sqrt{17}$

$\mathrm{Re}(-4+\mathrm{i}) < 0$ so $\arg(-4+\mathrm{i}) = \pi + \arctan\left(\frac{1}{-4}\right) = 2.90$

b $\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}$

c If points P and Q represent complex values p and q, respectively,

$p = -4+\mathrm{i}$

$$q = p\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}$$

$$= (-4+\mathrm{i})\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)$$

$$= -2\sqrt{3}-\frac{1}{2}+\left(\frac{\sqrt{3}}{2}-2\right)\mathrm{i}$$

Q has coordinates $\left(-2\sqrt{3}-\frac{1}{2}, \frac{\sqrt{3}}{2}-2\right) \approx (-3.96, -1.13)$

23

Eigenvalues are λ such that $\det\begin{pmatrix}2-\lambda & 1\\ 5 & 2-\lambda\end{pmatrix} = 0$

$4-4\lambda+\lambda^2-5=0$

$\lambda^2-4\lambda-1=0$

$\lambda = 2\pm\mathrm{i}\sqrt{5}$

24

$2|z| = |z+3|$

$2\sqrt{x^2+y^2} = \sqrt{(x+3)^2+y^2}$

$4x^2+4y^2 = x^2+6x+9+y^2$

$3x^2+3y^2-6x = 9$

$x^2+y^2-2x = 3$

Comment:

This form is fine as a final answer, since the question asks for the relationship between the variables.

You could continue to complete the square

$(x-1)^2+y^2 = 4$

or find y in terms of x

$y = \pm\sqrt{4-(x-1)^2}$

but this is not necessary to answer the question.

25 a

$$zw = (1+\text{i})(1+\text{i}\sqrt{3})$$
$$= (1-\sqrt{3}) + \text{i}(1+\sqrt{3})$$

b $|z| = \sqrt{1^2+1^2} = \sqrt{2}$

Since $\text{Re}\,(z) > 0, \arg z = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \text{i}\sin\left(\frac{\pi}{4}\right)\right)$$

$$|w| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$$

Since $\text{Re}\,(w) > 0, \arg w = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$$w = 2\left(\cos\left(\frac{\pi}{3}\right) + \text{i}\sin\left(\frac{\pi}{3}\right)\right)$$

Then $|zw| = 2\sqrt{2}$ and $\arg(zw) = \frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$

c Comparing the results from parts **a** and **b**

$$\sin\left(\frac{7\pi}{12}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

26 a $\left|\sqrt{3}+\text{i}\right| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$

$\text{Re}\,\left(\sqrt{3}+\text{i}\right) > 0$ so $\arg\left(\sqrt{3}+\text{i}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

b $\left(\sqrt{3}+\text{i}\right)^7 = \left(2\,\text{cis}\left(\frac{\pi}{6}\right)\right)^7 = 2^7\,\text{cis}\left(\frac{7\pi}{6}\right)$

$$\left(\sqrt{3}-\text{i}\right)^7 = \left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^7 = 2^7\,\text{cis}\left(-\frac{7\pi}{6}\right)$$

$$\left(\sqrt{3}+\text{i}\right)^7 + \left(\sqrt{3}-\text{i}\right)^7 = 2^7\left(2\cos\left(\frac{7\pi}{6}\right)\right) = 128 \times 2 \times \left(-\frac{\sqrt{3}}{2}\right) = -128\sqrt{3}$$

27 $z = 6 + 8\text{i}$ so $|z| = \sqrt{6^2+8^2} = 10$

$|w| = 5$

Let $w = u + \text{i}v$ for some real u, v

Then $u^2 + v^2 = |w|^2 = 25$

$$|z+w| = \sqrt{(6+u)^2 + (8+v)^2} = \sqrt{100 + u^2 + v^2 + 12u + 16v}$$

$$|z+w| = |z| + |w|$$

$$\sqrt{100 + u^2 + v^2 + 12u + 16v} = 15$$

Squaring: $100 + (u^2+v^2) + 12u + 16v = 225$

Substituting $u^2+v^2=25$: $125+12u+16v=225$

$3u+4v=25$

$3u=25-4v \qquad (*)$

But $u^2=25-v^2$

$9u^2=225-9v^2$

But $(*)$ gives $9u^2=625-200v+16v^2$

Substituting: $25v^2+200v-600=0$

$v^2+8v-48=0$

$(v-4)(v+12)=0$

$v=4$ (reject $v=-12$ since so $|v|\le|w|=5$

$u=3, v=4$ so $w=3+4\text{i}$

Tip:

This algebraic approach is laborious; it would be much faster to use the geometry of the complex plane.

Let $q=w+z$ and consider the points Q, W and Z, the representations of q, w and z on the Argand plane

The statement $|q|=|w|+|z|$ is the same as saying that OQ is the sum of OW and OZ, but this is the edge case of the triangle inequality, and can only occur when W and Z both lie on line segment OQ – that is, W and Z must be collinear (and the same side of the origin).

Thus $w=kz$ for some positive k. Since the question gives that $|w|=\frac{1}{2}|z|$, it follows that $w=\dfrac{1}{2}z=3+4\text{i}$

28 Let a and b be the complex values represented by A and B on the Argand plane.

$a=1+3\text{i}, b=-6+4\text{i}$

$|a|=\sqrt{1^2+3^2}=\sqrt{10}, |b|=\sqrt{6^2+4^2}=2\sqrt{13}$

$\text{Re}\,(a)>0$ so $\arg(a)=\arctan\left(\dfrac{3}{1}\right)=1.25$

$\text{Re}\,(b)<0$ so $\arg(b)=\pi+\arctan\left(\dfrac{4}{-6}\right)=2.55$

Then the transformation is an enlargement with scale factor $\frac{2\sqrt{13}}{\sqrt{10}}\approx 2.28$ and a rotation $2.55-1.25=1.30$ radians anticlockwise.

29 a

$$\begin{aligned} AB &= |z_1 - z_2| \\ &= \left|1 - \left(2 - \sqrt{3}\right)\mathrm{i}\right| \\ &= \sqrt{1^2 + \left(2 - \sqrt{3}\right)^2} \\ &= \sqrt{8 - 4\sqrt{3}} \\ &= 2\sqrt{2 - \sqrt{3}} \end{aligned}$$

b

$$\arg z_1 = \arctan\left(\frac{-2}{2}\right) = -\frac{\pi}{4}$$

$$\arg z_2 = \arctan\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

Then angle subtended at the origin between the two points in the complex plane is the difference in their arguments:

$$\arg z_1 - \arg z_2 = \frac{\pi}{12}$$

7 Graphs and algorithms

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 7A

13 a

$$\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

b C has degree 4, which indicates that C is friends with 4 of the other 6 in the group.

c A graph is complete if every vertex is connected directly to every other vertex. In this context, complete would mean that each of the seven was friends with every other member of the group.

d 8 additional edges are needed to make the graph complete.

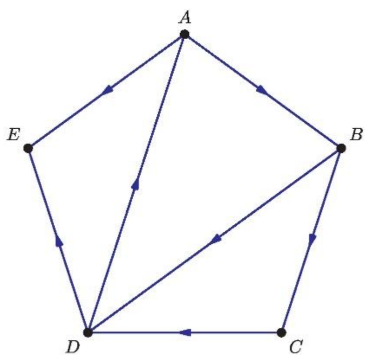
This can be seen from the total number of zeros in the above matrix being 23. 7 of these are the entries linking each point to itself, and the remaining 16 is the double count of the missing edges.

14 a A tree is a graph containing no closed paths; since $ABDEA$ forms a loop, this is not a tree.

b The graph indicates power cables; the graph must be connected if all the devices are to be on the same power network.

c Any of the four edges in the closed loop could be removed and the graph still be connected; AB, AE, BD or DE

15 a



b It is not possible to get from E to any other vertex.

c In a strongly connected graph, any vertex can be reached from any other.

d If either of the roads leading to E (AE or DE) were to be made two-way (or even just one-way in the other direction) then the graph would become strongly connected.

Exercise 7B

11 a Four of the vertices have degree 3 (B, D, F, H) and in an Eulerian graph all vertices have even degree.

b $SMNPQNQMRS$

Tip: There are many many possible examples. An example is given, but an exhaustive list is not worthwhile!

12 A semi-Eulerian graph has exactly two vertices of odd degree.

There are two vertices (Y, Z) with degree 3. The remainder have either degree 2 (U, X) or 4 (V, W).

Any trail must start at one of Y or Z and end at the other.

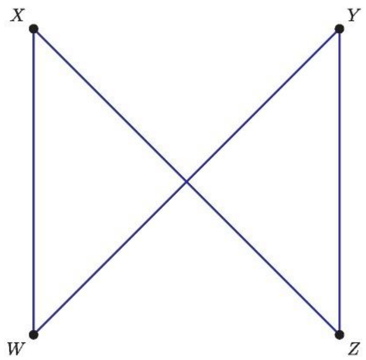
One circuit is $YWZVXWUVYZ$

13 a In A the first and fourth vertex have odd degree so A is not Eulerian

In B the vertices all have even degree and the graph is connected so it is Eulerian.

In C the vertices all have even degree and the graph is connected so it is Eulerian.

b



c Each step of the path moves either to the top row or the bottom row of the diagram, so each step except the last has two possible choices.

$2^4 = 16$

14 a A graph is Hamiltonian if it contains a Hamiltonian cycle, which visits each vertex exactly once, returning to the initial vertex without repeating an edge.

A possible Hamiltonian cycle is $ABEDCA$

b If the adjacency matrix is M then the number of walks of length 5 can be seen by examining M^5

$$M^5 = \begin{pmatrix} 56 & 72 & 72 & 56 & 80 \\ 72 & 56 & 56 & 72 & 80 \\ 72 & 56 & 56 & 72 & 80 \\ 56 & 72 & 72 & 56 & 80 \\ 80 & 80 & 80 & 80 & 96 \end{pmatrix}$$

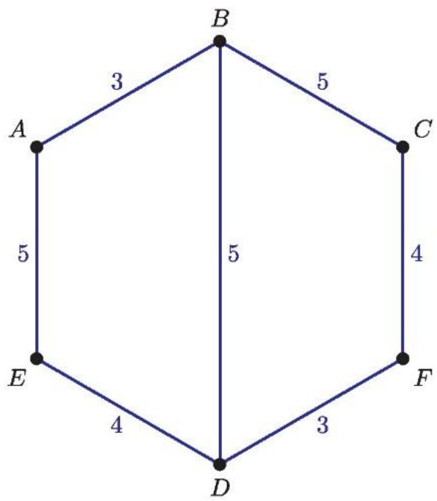
Hence there are 80 walks of length 5 from A to E.

Exercise 7C

11 a The shortest distance between B and C is 15000 km, flying via city A.

b The cheapest route costs \$350, flying via city E.

12 a



b The fastest walk from A to D is path ABD, which takes 8 minutes.

13 a $SCBDT$ is the shortest path from S to T, with a distance of 12 km.

b $SCBT$ or $SCDT$ are now the shortest, at 13 km.

14 a

$$M = \begin{pmatrix} 0 & \frac{1}{4} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & \frac{1}{3} & 0 \end{pmatrix}$$

b From the GDC, the entry from E to C in M^7 is 0.145, so this is the probability of visiting C on day 7.

c Taking a high power of M, the distribution of locations settles to $\mathrm{P}(B) \approx 33.3\%$ being the most likely.

15 The transition matrix for the five sites, assuming random selection from available choices is

$$M = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Taking a high power of M, the distribution of probabilities of being at each site settles to

$A: 0.293$

$B: 0.146$

$C: 0.195$

$D: 0.146$

$E: 0.220$

Ranking these, A is the most 'important', followed in order by E, C, B and D, with the last two equal.

16 a The transition matrix for the four sites, assuming random travel between them from available choices is

$$M = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Taking a high power of M, the distribution of probabilities of being at each site settles to

$A: 0.129$

$B: 0.194$

$C: 0.290$

$D: 0.387$

The most popular site is therefore D.

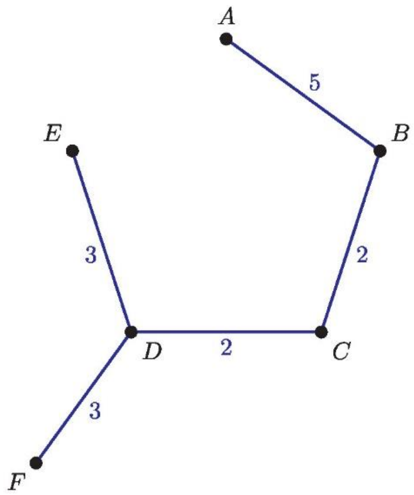
b The new transition matrix Q with E meaning the search ends is

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0.3 & 0 \\ 0.45 & 0 & 0 & 0.3 & 0 \\ 0.45 & 0.45 & 0 & 0.3 & 0 \\ 0 & 0.45 & 0.9 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.1 & 1 \end{pmatrix}$$

c The relative probabilities of each site remains the same. Q to a large power inevitably gives all probability to E, indicating that every walk through the sites eventually ends. Since the probability of terminating a walk is the same at every location, their relative probabilities will be unchanged, however.

Exercise 7D

13 a



b 15 km

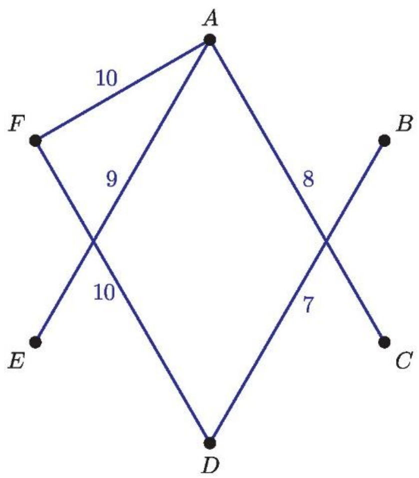
14 a BD, CD, CE, AB and AF or EF

b Start with CF then apply the usual algorithm, adding further edges in order of increasing travel time unless a loop is formed until all vertices are connected.

15 a Kruskal's algorithm adds one edge at a time, and during the process there may be multiple disconnected subgraphs.

Prim's algorithm adds one vertex at a time, and at all stages of the process there is a single, growing, connected graph.

b



c 44 m

d i The graph is connected, so each workstation is connected to the power supply

ii The graph is a tree, so each workstation has only one single connection path to the power supply, there is no redundancy or loops.

16 **a** The 5 vertices can be connected using 4 edges.

i The minimum spanning tree is therefore $10 + 11 + 13 + 15 = 49$

ii Using Kruskal's algorithm in the worst case scenario:

Use 10

Use 11 (cannot be duplicating 10 since the graph is simple)

Reject 13 (loop)

Use 15 (cannot also make a loop since the graph is simple)

Reject 18 (loop)

Reject 20 (loop)

Use 22 as the fourth edge

The maximum spanning tree is therefore $10 + 11 + 15 + 22 = 58$

17 Using Prim's algorithm, the minimum spanning tree starting at the base vertex is

$6 + 9 + 12 + x + 6$ if $x \leq 28 - x$

$28 - x + 6$ if $x \geq 28 - x$

So the total is $33 + x$ if $x \leq 14$

The total is $61 - x$ if $x \geq 14$

If the minimum spanning tree is less than 40 then

If $x \leq 14$, $x < 7$

If $x \geq 14, x > 21$

But $x < 28$ since all edge lengths are positive.

So $0 < x < 7$ or $21 < x < 28$

Exercise 7E

5 **a** O and C have degree 3

b 9 (path CBO)

c The total length of all edges is 72. With the addition of the repeated path, the shortest route is 81.

d OB and BC

6 **a** His start and end points must be F and D. The total distance of the route is 6800 m.

b F and D have odd degrees, so it is not possible to take each street once if not starting at either of these, finishing at the other.

c The shortest path from D to F is 12.5 (DEF) so the shortest postal route starting and ending at A is $6800 + 1250 = 8050$ m

d DE and EF

7 B and F have odd degree, so a path between them must be travelled twice. Shortest such path is BDF, of length 0.9 km.

Total of all road lengths is 7.8 km so the minimal path to travel each road at least once is 8.7 km

$CBAJIHGFEDBDFHJCFDC$

8 **a** I_2 and SB each have odd degree, so a route moving along each edge at least once must either start at one and end at the other or must duplicate some path between them.

The total of all the times is 86 and the shortest path between I_1 and SB is 13 which must be travelled twice.

The shortest time is therefore 99 minutes.

A possible route is NB $-$ I_1 $-$ SB $-$ I_1 $-$ I_2 $-$ SB $-$ I_1 $-$ NB:

$13 + 13 + 17 + 12 + 15 + 13^* + 16 = 99$

(13^* indicating the repeated bridge crossing)

b Now every vertex has odd degree, so the duplicated journeys must be grouped in two pairs and minimised.

NB $-$ I_1 (13) and SB $-$ I_2(15) for a total 28 duplicated will be minimal.

The new total for all roads is $86 + 16 = 102$ minutes

The new minimum route taking all bridges and starting and ending at the North bank is

$102 + 28 = 130$ minutes

For example: NB $-$ I_1 $-$ NB $-$ I_2 $-$ SB $-$ I_2 $-$ I_1 $-$ SB $-$ I_1 $-$ NB

$13 + 16 + 16 + 15 + 15^* + 12 + 13 + 17 + 13^* = 130$

9 **a** Vertices A, D, E and H all have odd degree; considering these in pairs, the most efficient route will repeat AD and EH.

b The total time of all paths is 93. With the repeated paths, this makes a total shortest route time of $93 + 6 + 10 = 109$ minutes

For example: $ABCADCFGHFEHEDA$

c The tourist can make a route only repeating DE, for a total of $93 + 7 = 100$ minutes.

d For example: $ABCADCFEDEHFGH$

10 **a** Vertices D, F, G and H all have odd degrees, which will require repeated routes to cover all paths. The most efficient paths linking these in pairs are FG and DH

The total of all paths is 183 m

So the shortest possible route from A to A covering all paths is $183 + 12 + 13 = 208$ m

For example: $AFGFEGBHEDHDCBA$

b FG and DH

c Starting at D and ending at F would require a path between G and H to be repeated, the shortest of which is GEH for 25 m.

The shortest such route is therefore $183 + 25 = 208$ m

For example:

$DEFABCDHBGEHEGF$

d Now only D and F have odd degrees and would need a path repeated; the shortest is $DEF = 31$

The total of all paths including the new GH is now 199 m

The shortest possible route is therefore $199 + 31 = 230$ m

For example:

$ABCDEFGEHBGHDEFA$

11 a In an Eulerian graph, all vertices have even degrees; C and F have odd degrees (5 and 3, respectively), so this is not Eulerian.

b CEF has weight 15

c The route will need to repeat CEF. The total of all the paths is 165 m

The optimal route covering all paths at least once is therefore $165 + 15 = 180$ m

For example: $DEFDBACBECFECD$

d i The shortest such route will be 165

For example:

$FEDFCABCDBEC$

ii Starting and ending vertices would need to be the odd degree vértices C and F.

Exercise 7F

10 a $ABCDA, ABDCA, ACBDA$

b Totalling each of these three possibilities, the shortest is $ABDCA$ (45)

11 a $AFEBCDA$: 32

$AFEDCBA$: 37

The nearest neighbour algorithm from A gives an upper bound of 32.

b $BFEADCB$: 31

c It is at most 31 km long

12 a Starting at A and using a nearest neighbour algorithm, path $ACBEDA$ gives an upper bound to route length of 29.

b Starting at B: $BCAEDB$ has length 30

Starting at C: $CABEDC$ has length 31, $CBAEDC$ has length 30

Starting at D: $DACBED$ has length 29

Starting at E: $ECABDE$ has length 30, $ECBADE$ has length 28

An improved upper bound is 28 minutes.

c Without D, a spanning tree for the vertices $ABCE$, using Kruskal's algrorithm, is CA, CB, CE for a total 11 minutes.

The two shortest paths linking D to this tree are DA and DC, for a total of 15.

The lower bound is therefore 26 minutes.

d The shortest possible required time is $24, 25, 26, 27$ or 28 minutes.

13 **a** $PUSR$ has a total distance $14 + 10 + 16 = 40$ km.

b PUQ has total distance $14 + 11 = 25$ km.

c

	P	***Q***	***R***	***S***	***T***	***U***
P	–	**25**	40	**24**	**26**	14
Q	**25**	–	20	21	23	11
R	40	20	–	**16**	**28**	26
S	**24**	21	**16**	–	**12**	10
T	**26**	23	**28**	**12**	–	12
U	14	11	26	10	12	–

d Route $QUSTPRQ$ has total distance $11 + 10 + 12 + 26 + 40 + 20 = 119$ km

e Spanning tree for $PRSTU$ using Kruskal's algorithm:

$US(10)$; UT or $ST(12)$; $UP(14), SR(16)$ has total 52

Then the shortest two paths connecting Q into the tree are $QU(11)$ and $QR(20)$.

Adding these to the total, a lower bound for the shortest possible route is 83 km.

14 **a** Route $ABCDEFA$ has total cost $5 + 6 + 7 + 9 + 6 + 7 = 40$

b Spanning tree for $ACDEF$ using Kruskal's algorithm:

$EF(6), CD(7), AF(7), DE(9)$ has a total cost 29.

Linking B in using its two shortest paths $AB(5)$ and $BC(6)$ gives a lower bound of 40 for a route.

c Since the upper and lower bounds are equal, the weight of the optimal route is exactly 40.

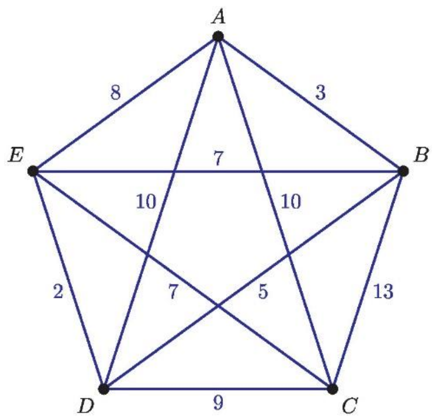
d The route from part **a** is one possible such optimal route: $ABCDEFA$.

15 a i $BDE(7)$

ii $BAC(13)$

iii $CED(9)$

b



c Nearest neighbour gives route $CEDBAC$ at a total time of $7 + 2 + 5 + 3 + 10 = 27$ minutes

d Spanning tree for $ABDE$, using Kruskal's algorithm:

$DE(2), AB(3), BD(5)$ for a total of 10.

Adding in the shortest two paths from C: $CE(7)$ and $CD(9)$ gives a total 26 minutes.

e Any route beginning and ending at C must use two of the paths from C.

Options:

7 and 7 (CE twice): There is no route for $ABDE$ of length 12 beginning and ending at E; the four shortest path lengths sum to 18.

7 and 9 (CE, CD): This is another version of CE twice, since the optimal route to D passes through E.

7 and 10 (CE, CA): This links to either end of the spanning tree $ABDE$, so is achievable as a route.

Every other option leads to linking routes with a total greater than 17, so cannot be an improvement since the spanning tree of $ABDE$ cannot be improved upon.

f 27 minutes: $CEDBAC$.

Mixed Exercise 7

1 a Second network; all vertices have even degree.

b Several possible routes; one example is $ABCDEBDA$.

c He would need to start at a vertex with an odd degree, C or E.

2 a Removing RS would leave every vertex with an even degree, making the graph Eulerian.

b Several possible examples, such as $PQRSUV$.

c Several possible examples, such as $PQRSVUP$.

3 **a**

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

b

$$M^5 = \begin{pmatrix} 4 & 10 & 14 & 26 & 20 \\ 10 & 18 & 22 & 34 & 26 \\ 14 & 22 & 16 & 22 & 14 \\ 26 & 34 & 22 & 18 & 10 \\ 20 & 26 & 14 & 10 & 4 \end{pmatrix}$$

There are 20 walks of length 5 from A to E

$$M^4 = \begin{pmatrix} 9 & 11 & 6 & 3 & 1 \\ 11 & 15 & 8 & 7 & 3 \\ 6 & 8 & 8 & 8 & 6 \\ 3 & 7 & 8 & 15 & 11 \\ 1 & 3 & 6 & 11 & 9 \end{pmatrix}$$

There are 7 walks of length 4 between B and D

4 **a** A tree of minimum weight which includes every vertex of the graph.

b Kruskal's algorithm: Beginning with the lowest weight edge, include paths in increasing order without forming a loop, until all vertices are connected.

BH, NF, HN, HA, BE, CN; the total weight of the tree is 48.

c The paths indicated in part **b**: BH, NF, HN, HA, BE, CN

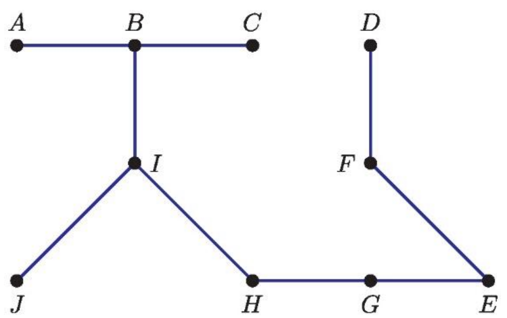
5 **a** Kruskal's algorithm adds edges, in order of increasing weight, ignoring edges which would form a loop, until all vertices are included in the tree. During the process, there may be several unconnected trees.

Prim's algorithm adds vertices, each being selected as requiring minimum weight to join it to the growing tree. During the process, there will always be a single connected tree.

b The tree grows using the following edges:

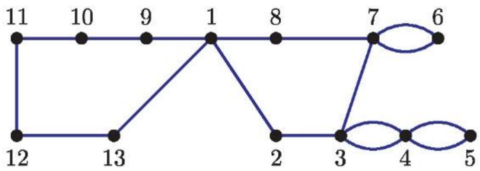
$AB, BC, BI, IJ, IH, HG, GE, EF, FD$

The tree produced is



The weight of the tree is 117.

6 **a** **i**



ii The following rooms have degree 2: 2, 5, 6, 8, 9, 10, 11, 12, 13

The remaining rooms have degree 4: 1, 3, 4, 7

iii All vertices have even degree, so it is possible to form a route passing through each doorway (ie using each edge of the graph) exactly once, finishing in the same room (1) as the route begins.

b **i** Graph G has 15 edges, graph H has 22.

ii Graph G has an Eulerian trail; exactly two vertices (E and F) have odd degrees.

In graph H there are four vertices (A, B, D and E) which have odd degree, so no such trail is possible.

iii Both graphs have vertices of odd degree, so an Eulerian circuit is not possible.

7 **a** The teacher needs a Hamiltonian cycle, visiting each vertex once.

b An upper bound is 111, using path $BAFCDEB$

c Removing B, a path of length 73 can be found ($CF(17), ED(18), CD(19), AF(19)$).

The minimal distance of two edges to join B to the remaining vertices is 33 ($AB(16), BC(17)$), so a lower bound for the circuit length for the full graph is $73 + 33 = 106$

d $106 \leq T \leq 111$

8 **a** $n - 1$

b **i** Edges added: AB, BC, AD, DE for a total 12 m

ii 12 m

c **i** Every vertex has even degree.

ii $ABCDEACEBDA$, which has length 46.

9 The transition matrix for this network is

$$T = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \end{pmatrix}$$

Each column of T^{50} has the form $\begin{pmatrix} 0.098 \\ 0.171 \\ 0.293 \\ 0.195 \\ 0.244 \end{pmatrix}$

In order of importance, the pages are $C(0.293), E(0.244), D(0.195), B(0.171), A(0.098)$

10 Without A: Minimum tree for $BCDE$ using nearest neighbour is $CD(54), BC(58), DE(61)$

Total is 173 and A can be joined to the tree by $AB(41)$ and $AE(62)$ for a lower bound of

$173 + 103 = 276$

Without B: Minimum tree for $ACDE$ using nearest neighbour is $CD(54), DE(61), AE(62)$

Total is 177 and B can be joined to the tree by $AB(41)$ and $BC(58)$ for a lower bound of

$177 + 99 = 276$

Without C: Minimum tree for $ABDE$ using nearest neighbour is $AB(41), DE(61), AE(62)$

Total is 161 and C can be joined to the tree by $CD(54)$ and $BC(58)$ for a lower bound of

$164 + 112 = 276$

Without D: Minimum tree for $ABCE$ using nearest neighbour is $AB(41), BC(58), AE(62)$

Total is 161 and D can be joined to the tree by $CD(54)$ and $DE(61)$ for a lower bound of

$161 + 115 = 276$

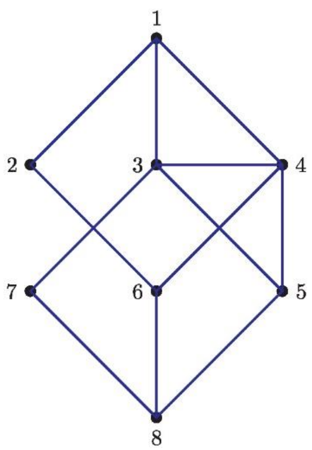
Without E: Minimum tree for $ABCD$ using nearest neighbour is $AB(41)$, $CD(54), BC(58)$

Total is 161 and E can be joined to the tree by $DE(61)$ and $AE(62)$ for a lower bound of

$153 + 123 = 276$

Lower bound for the travelling salesman problem is 276

11 **a**



b Vertices 1 and 8 have odd degrees.

c

$$T = \begin{pmatrix} 0 & 1/2 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 1/3 & 0 & 0 & 1/4 & 1/4 & 0 & 1/2 & 0 \\ 1/3 & 0 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 & 0 & 1/4 & 0 & 1/3 \\ 0 & 1/2 & 0 & 1/4 & 1/4 & 0 & 0 & 1/3 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/2 & 0 \end{pmatrix}$$

d The element in the first row, third column of T^3 is 0.215; this is the probability that, given a start in Room 1, the third room entered is Room 3.

e Each column of T^{50} is approximately $\begin{pmatrix} 0.115 \\ 0.077 \\ 0.154 \\ 0.154 \\ 0.154 \\ 0.154 \\ 0.077 \\ 0.115 \end{pmatrix}$, so on a random journey, approximately 11.5% of the time would be spent in Room 8.

12 a A and E both have odd degrees.

b The shortest path between A and E is AGE, with length 35. This will have to be repeated in such a route.

The length of the route is then the total length of the paths plus this repeated section

$(35 + 40 + 15 + 15 + 20 + 20 + 20 + 40 + 10) + 35 = 250\,\text{m}$

Such a path might be $ABCADEFAGEGA$.

c A path between A and E directly would allow a route along each path exactly once, eliminating the need for the repeated journey.

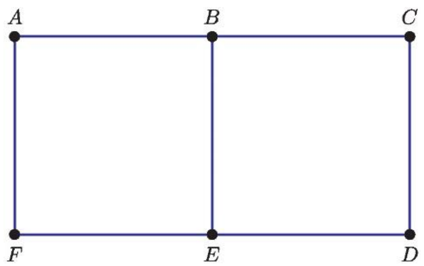
13 a i Any path beginning and ending at the same vertex, using all edges at least once: For example, $PQRSTSRTQP$

ii 34

b i To find a Hamiltonian cycle of least weight.

ii There is no cycle containing vertex P.

14 a



b The number of walks of length 2 from each vertex back to itself.

c

$$\mathbf{M}^4 = \begin{pmatrix} 9 & 0 & 8 & 0 & 12 & 0 \\ 0 & 17 & 0 & 12 & 0 & 12 \\ 8 & 0 & 9 & 0 & 12 & 0 \\ 0 & 12 & 0 & 9 & 0 & 8 \\ 12 & 0 & 12 & 8 & 17 & 0 \\ 0 & 12 & 0 & 0 & 0 & 9 \end{pmatrix}$$

So there are 8 walks from A to C of length 4.

d $ABEDC, AFEBC, AFEDC$

(the other walks involve walking back and forth on a side path from the direct route ABC, and so are not trails: $ABABC, AFABC, ABEBC, ABCBC, ABCDC$)

15 a Without A: Minimum tree for $BCDE$ using nearest neighbour is $BC(11), BE(13), CD(14)$

Total is 38 and A can be joined to the tree by $AD(10)$ and $AE(15)$ for a lower bound of

$38 + 25 = 63$

b $ADCBEA$: 63

c Since the lower bound is achievable using the cycle suggested, this cycle is a solution to the travelling salesman problem for this graph.

16 a i There are vertices (A, B, C, D, E, F) of odd degree

ii This is also not possible, since there are more than 2 vertices with odd degree.

b i All the vertex degrees have doubled, so all are now even.

ii Summing all the edges of the graph and doubling: 306 km

17 The vertices with odd degrees are A, B, C and D; paired up, these will need to have a connecting route repeated in the cycle

The costs of the paired routes:

$A - B$ and $C - D$: AEB (11), CGD (8)

$A - C$ and $B - D$: $AEFGC$ (13), $BEFGD$(14)

$A - D$ and $B - C$: $AD(10)$ and $BC(10)$

The lowest of these is $A - B$ and $C - D$, which uses edges AE, EB, DG and GC

The total cost is then the total of all the edges, with these four repeated: $92

18 a

	A	B	C	D	E	F
A	–	8	8	16	18	18
B	8	–	7	15	17	17
C	8	7	–	8	10	10
D	16	15	8	–	6	8
E	18	17	10	6	–	0
F	18	17	10	8	9	–

b Without B: Minimum spanning tree for $ACDEF$ using nearest neighbour is $DE(6), AC(8), CD(8), DF(8)$

Total is 30 and B can be joined to the tree by $BC(7)$ and $AB(8)$ for a lower bound of

$30 + 15 = 45$

c Using nearest neighbour, the path given is $CB(7), BA(8), AD(16), DE(6), EF(9), FC(10)$, which gives an upper bound of 56.

d $45 \leq T \leq 56$

e $CBACDEFC$

19 a CD, AB, BC, BF, CE

b i A connects minimally through AB and AC for a total 77

Prim's algorithm shows that $BCDEF$ has minimum spanning tree $CD + BC + BF + CE = 154$

Lower bound is $154 + 77 = 231$

ii B connects minimally through AB and BC for a total 74

Prim's algorithm shows that $ACDEF$ has minimum spanning tree $CD + AC + CE + EF = 165$

Lower bound is $165 + 74 = 239$

iii The greater value is a better lower value: 239

c $ABCDEFA$ has length 253 which is clearly a valid upper bound

The optimal solution L satisfies the inequality $239 \leq L \leq 253$.

20 a Deleting A:

A connects minimally through AB and AE for a total 13

Prim's algorithm shows that $BCDE$ has minimum spanning tree $DE + CD + BD$ or $BE = 19$

Lower bound is $19 + 13 = 32$

Deleting B:

B connects minimally through AB and BD or BE for a total 14

Prim's algorithm shows that $ACDE$ has minimum spanning tree $DE + CD + AE = 18$

Lower bound is $18 + 14 = 32$

Deleting C:

C connects minimally through BC and CD for a total 15

Prim's algorithm shows that $ABDE$ has minimum spanning tree $DE + AE + AB = 18$

Lower bound is $18 + 15 = 33$

Deleting D:

D connects minimally through CD and DE for a total 11

Prim's algorithm shows that $ABCE$ has minimum spanning tree $AB + BE + BC = 22$

Lower bound is $22 + 11 = 33$

Deleting E:

E connects minimally through AE and DE for a total 12

Prim's algorithm shows that $ABCD$ has minimum spanning tree $AB + BD + CD = 20$

Lower bound is $20 + 13 = 32$

The best lower bound from this method is 33

b Since the cycle $ACBDEA$ has length 33, it is clear that this lower bound is achievable; since 33 is a lower bound, it is also known that this length cannot be improved upon.

8 Probability

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 8A

16 a

$$\begin{aligned}\mathrm{E}(X) &= \sum_x x \times \mathrm{P}(X = x)\\ &= (1 \times 0.2) + (3 \times 0.3) + (5 \times 0.2) + (7 \times 0.3)\\ &= 4.2\end{aligned}$$

b $\mathrm{E}(3X + 1) = 3\mathrm{E}(X) + 1 = 13.6$

c $\mathrm{Var}(3X + 1) = 9\mathrm{Var}(X) = 44.64$

17 a

$$\begin{aligned}1 &= \sum_w w \times \mathrm{P}(W = w)\\ &= k\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)\\ &= \frac{15}{8}\end{aligned}$$

$$k = \frac{8}{15}$$

b $\mathrm{Var}(2W - 2) = 4\mathrm{Var}(W) = 13.8$

18 a

$$\begin{aligned}\mathrm{E}(X) &= \sum_x x \times \mathrm{P}(X = x)\\ &= \left(1 \times \frac{1}{16}\right) + \left(2 \times \frac{3}{16}\right) + \left(3 \times \frac{5}{16}\right) + \left(4 \times \frac{7}{16}\right)\\ &= \frac{25}{8}\end{aligned}$$

b $\mathrm{E}(10X + 3) = 10\mathrm{E}(X) + 3 = \frac{137}{4}$

c $\mathrm{Var}(10X + 3) = 100\mathrm{Var}(X) = 85.9$

19 a

$$\begin{aligned}\mathrm{E}(2X - Y + 4) &= 2\mathrm{E}(X) - \mathrm{E}(Y) + 4\\ &= 15\end{aligned}$$

b

$$\text{Var}(2X - Y + 4) = 4\text{Var}(X) + \text{Var}(Y)$$
$$= 17$$

20 Let A, B, C be the thickness of the bun base, burger and bun top respectively, in cm.

$\text{E}(A) = 1.4, \text{E}(B) = 3.0, \text{E}(C) = 2.2$

$\text{Var}(A) = 0.02, \text{Var}(B) = 0.14, \text{Var}(C) = 0.2$

$\text{E}(A + B + C) = \text{E}(A) + \text{E}(B) + \text{E}(C) = 6.6\text{ cm}$

$\text{Var}(A + B + C) = \text{Var}(A) + \text{Var}(B) + \text{Var}(C) = 0.36$

$\text{SD}(A + B + C) = \sqrt{\text{Var}(A + B + C)} = 0.6\text{ cm}$

21 Let X be the mass of an egg. $\text{E}(X_i) = 12.4, \text{Var}(X_i) = 1.2^2$

Then Y, the mass of a box of twelve eggs, is given by

$$Y = 50 + \sum_1^{12} X_i$$

$\text{E}(Y) = 50 + 12\text{E}(X_i) = 198.8\text{ g}$

$\text{Var}(Y) = 12\text{Var}(X_i) = 17.28$

$\text{SD}(Y) = \sqrt{\text{Var}(Y)} = 4.16\text{ g}$

22 Let X_i be the mass of a chocolate bar in grams.

$\text{E}(X_i) = 102, \text{Var}(X_i) = 8.6^2$

Let Y be the mean of twenty randomly sampled chocolate bars (assuming their masses are independent).

$\text{E}(Y) = \text{E}(X_i) = 102\text{ g}$

$$\text{SD}(Y) = \frac{\text{SD}(X)}{\sqrt{20}} = 1.92\text{ g}$$

23 $B = 5 - \frac{1}{2}A$

$$\text{E}(B) = 5 - \frac{1}{2}\text{E}(A) = 3.1$$

$$\text{Var}(B) = \frac{1}{4}\text{Var}(A) = 0.3$$

24

$$V = 4 - \frac{2}{5}U$$

$$\text{E}(V) = 4 - \frac{2}{5}\text{E}(U) = -6$$

$$\text{Var}(V) = \frac{4}{25}\text{Var}(U) = 2.56$$

25 Let X_i be the mass of a man in the office, and Y_i be the mass of a woman, in kg.

$\mathrm{E}(X) = 78, \mathrm{E}(Y) = 62$

$\mathrm{Var}(X) = 12^2, \mathrm{Var}(Y) = 8^2$

The mass of the lift containing 3 women and 5 men is L

$$L = 500 + \sum_1^5 X_i + \sum_1^3 Y_i$$

$\mathrm{E}(L) = 500 + 5\mathrm{E}(X) + 3\mathrm{E}(Y) = 1076$ kg

$\mathrm{Var}(L) = 5\mathrm{Var}(X) + 3\mathrm{Var}(Y) = 5328$

$\mathrm{SD}(L) = \sqrt{\mathrm{Var}(L)} = 30.2$ kg

26 Let A be Alain's roll and B_i be Beatrice's ith roll.

$\mathrm{E}(A) = \mathrm{E}(B) = 4$

$\mathrm{Var}(A) = \mathrm{Var}(B) = 1$

Let $C = 2A - \sum_1^2 B_i$

$\mathrm{E}(C) = 2\mathrm{E}(A) - 2\mathrm{E}(B) = 0$

$\mathrm{Var}(C) = 4\mathrm{Var}(A) - 2\mathrm{Var}(B) = 2$

$\mathrm{SD}(C) = \sqrt{\mathrm{Var}(C)} = 1.41$

27 Let X_i be the score of the ith student

$\mathrm{E}(X_1 - X_2) = \mathrm{E}(X_1) - \mathrm{E}(X_2) = 0$

$\mathrm{Var}(X_1 - X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 1350$

$\mathrm{SD}(X_1 - X_2) = \sqrt{\mathrm{Var}(X_1 - X_2)} = 35.4$

28 If the standard deviation for a single cucumber is k then the standard deviation of the mean of two is $\frac{k}{\sqrt{2}}$

$$\frac{k}{\sqrt{2}} = k - 20$$

$$k\left(1 - \frac{1}{\sqrt{2}}\right) = 20$$

$$k = \frac{20}{1 - \frac{1}{\sqrt{2}}} = 68.3 \text{ g}$$

29 Let A be the time one of Ayane's journeys and B be the time of one of Hiroki's, in minutes.

$\mathrm{E}(A) = 15, \mathrm{E}(B) = 28$

$\mathrm{Var}(3^2) = \mathrm{Var}(B) = 2^2$

Let $C = \sum_1^5 B_i - \sum_1^5 A_i$ be the amount of additional time Hiroki takes in a week

$\mathrm{E}(C) = 5\mathrm{E}(B) - 5\mathrm{E}(A) = 65$ min

$\mathrm{Var}(C) = 5\mathrm{Var}(B) + 5\mathrm{Var}(A) = 65$

$\mathrm{SD}(C) = \sqrt{\mathrm{Var}(C)} = 8.06$ min

30 **a** Let the sample size be n, and the lifetime of a battery, in hours, be X_i

$\mathrm{E}(X) = 4.8, \mathrm{Var}(X) = 1.7$

Standard deviation of the sample mean is $\sqrt{\frac{1.7}{n}} < 0.3$

$\frac{1.7}{n} < 0.09$

$n > \frac{1.7}{0.09} = 18.9$

The sample size must be at least 19.

b

$\sqrt{\frac{1.7}{n+80}} = \frac{1}{3}\sqrt{\frac{1.7}{n}}$

$\frac{1.7}{n+80} = \frac{1}{9}\left(\frac{1.7}{n}\right)$

$9n = n + 80$

$8n = 80$

$n = 10$

The original sample size was 10.

Exercise 8B

9 **a** $\mathrm{E}(3X + 1 - 5Y) = 3\mathrm{E}(X) + 1 - 5\mathrm{E}(Y) = 36 + 1 - 40 = -3$

$\mathrm{Var}(3X + 1 - 5Y) = 9\mathrm{Var}(X) + 25\mathrm{Var}(Y) = 144 + 625 = 769$

$3X + 1 - 5Y \sim \mathrm{N}(-3, 769)$

$\mathrm{P}(3X + 1 - 5Y < 0) = 0.543$

b $\mathrm{E}(X_1 + X_2 - Y_1 + Y_2) = 2\mathrm{E}(X) = 24$

$\mathrm{Var}(X_1 + X_2 - Y_1 + Y_2) = 2\mathrm{Var}(X) + 2\mathrm{Var}(Y) = 32 + 50 = 82$

$X_1 + X_2 - (Y_1 - Y_2) \sim \mathrm{N}(24, 82)$

$\mathrm{P}(X_1 + X_2 - (Y_1 - Y_2) > 0) = 0.996$

10 For $n = 30, \bar{X} \sim \mathrm{N}\left(100, \frac{25^2}{30}\right)$

a $\mathrm{P}(\bar{X} < 98) = 0.331$

b $1 - \mathrm{P}(85 < \bar{X} < 105) = 0.137$

11 a The sum of two normal distributions is also a normal distribution.

Let X be the mass of a passenger and their hand luggage, in kg.

$X \sim \mathrm{N}(91.3, 16.3)$

b $\mathrm{P}(X > 100) = 0.0156$

12 a $A \sim \mathrm{N}(13.1, 0.4^2), B \sim \mathrm{N}(12.8, 0.6^2)$

$A - B \sim \mathrm{N}(0.3, 0.52)$

Mean is 0.3 s, standard deviation is $\sqrt{0.52} = 0.721$ s

b $\mathrm{P}(A - B < 0) = 0.339$

c $\mathrm{P}(A - B > 1) = 0.166$

13 Let X be the length, in cm, of one rod. $X \sim \mathrm{N}(65, 0.03)$

$n = 6$ so $\bar{X} \sim \mathrm{N}\left(65, \frac{0.03}{6}\right)$

a Mean of $\bar{X}$ is 65 cm

Variance of $\bar{X}$ is 0.005 cm^2

b $1 - \mathrm{P}(64.8 < \bar{X} < 65.3) = 0.00235$

14 Let X be the length of a pipe in cm.

$X \sim \mathrm{N}(40, 3^2)$

a $\mathrm{P}(X > 42) = 0.252$

b For $n = 10, \bar{X} \sim \mathrm{N}(40, 0.9)$

$\mathrm{P}(\bar{X} > 42) = 0.0175$

15 For $n = 40$, CLT gives that $\bar{X}$ has approximate distribution $\mathrm{N}\left(12, \frac{3.5^2}{40}\right)$

$\mathrm{P}(13 < \bar{X} < 14) = 0.0352$

16 Let X be the mass of a pineapple in grams. $\mathrm{E}(X) = 145, \mathrm{Var}(X) = 96$

By CLT, for $n = 70, \bar{X}$ has approximate distribution $\mathrm{N}\left(145, \frac{96}{70}\right)$

$\mathrm{P}\left(\bar{X} < \frac{10000}{70}\right) = 0.0336$

17 Let X be the number of calories Winnie eats in a day. $\mathrm{E}(X) = 1900, \mathrm{Var}(X) = 400$^2

By CLT, for $n = 31, \bar{X}$ has approximate distribution $\mathrm{N}\left(1900, \frac{160000}{31}\right)$

$\mathrm{P}(\bar{X} > 2000) = 0.0820$

18 $X \sim \mathrm{N}(4.6, 0.25^2), Y \sim \mathrm{N}(2.5, 0.2^2)$

a $2Y - X \sim \mathrm{N}(0.4, 0.2225)$

b $\mathrm{P}(2Y - X) < 0 = 0.198$

c

Let S

$= \sum_1^3 X_i + \sum_1^4 Y_i$ be the sum of three males and four females, randomly selected.

$S \sim \mathrm{N}(23.8, 0.3475)$

$\mathrm{P}(S > 25) = 0.0209$

19 Let A be the mass of an apple and B the mass of a pear, in grams.

$A \sim \mathrm{N}(180, 144), B \sim \mathrm{N}(100, 100)$

a $\mathrm{P}(A > 2B) = \mathrm{P}(A - 2B > 0)$

$A - 2B \sim \mathrm{N}(-20, 544)$

$\mathrm{P}(A - 2B > 0) = 0.196$

b $A_1 + A_2 + B \sim \mathrm{N}(460, 388)$

$\mathrm{P}(A_1 + A_2 + B > 500) = 0.0211$

20 Let X be the length of a grass snake, in metres. $X \sim \mathrm{N}(1.2, \sigma^2)$

For $n = 5, \bar{X} \sim \mathrm{N}\left(1.2, \frac{\sigma^2}{5}\right)$

$\mathrm{P}(\bar{X} > 1.4) = 0.05$

$1.4 = 1.2 + z_{0.95} \frac{\sigma}{\sqrt{5}} = 1.2 + 1.645 \left(\frac{\sigma}{\sqrt{5}}\right)$

$\sigma = \frac{\sqrt{5}}{1.654}(0.2) = 0.272 \text{ m}$

21 $\mathrm{E}(X) = 10 \times 0.6 = 6, \mathrm{Var}(X) = 10 \times 0.6 \times 0.4 = 2.4$

For $n = 35, \bar{X}$ has approximate distribution $\mathrm{N}\left(6, \frac{2.4}{35}\right)$

$\mathrm{P}(\bar{X} > 6.5) = 0.0281$

22 Let X be the mass, in grams, of a sheet of A4 paper.

$\mathrm{E}(X) = 5, \mathrm{Var}(X) = 0.08^2$

By CLT, for $n = 500, \bar{X}$ has approximate distribution $\mathrm{N}(5 \times 500, 0.08^2 \times 500)$

$\bar{X} \sim \mathrm{N}(2500, 3.2)$

a Mean is 2500 g, standard deviation is $\sqrt{3.2} = 1.79$ g

b $\mathrm{P}(2495 < \bar{X} < 2505) = 0.995$

c The distribution of the mass of an individual sheet is not known, but the CLT gives that the distribution of the mean or total of many sheets of paper will be approximately normal, no matter that distribution, as long as the variance is finite (which it is here).

23 Using CLT:

$\bar{A}_{40}$ has approximate distribution $\mathrm{N}\left(45, \frac{22^2}{40}\right)$

$\bar{B}_{50}$ has approximate distribution $\mathrm{N}\left(51, \frac{25^2}{50}\right)$

Then $\bar{A}_{40} - \bar{B}_{50} \sim \mathrm{N}\left(-6, \frac{22^2}{40} + \frac{25^2}{50}\right) = \mathrm{N}(-6, 24.6)$

a $\mathrm{P}(-4 < \bar{A}_{40} - \bar{B}_{50} < 4) = 0.321$

b It is not known what distribution the individual times to attend would follow, but using CLT we can be confident that the mean times are approximately normal for each company.

24 a Let B be the score of a boy and G the score of a girl.

$B - G \sim \mathrm{N}(-10, 41)$

$\mathrm{P}(-5 < B - G < 5) = 0.208$

b

$B - \frac{3}{4}G \sim \mathrm{N}\left(5, 25 + \frac{9}{16} \times 16\right) = \mathrm{N}(5, 34)$

$\mathrm{P}\left(B - \frac{3}{4}G < 0\right) = 0.196$

25 Let X be the amount of rainfall in a day.

$X \sim \mathrm{N}(\mu, \sigma^2)$

For $n = 7, \bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{7}\right)$

$\mathrm{P}(X > 8) = 0.1 \Rightarrow 8 = \mu + z_{0.9}\sigma = \mu + 1.282\sigma(1)$

$\mathrm{P}(\bar{X} < 7) = 0.05 \Rightarrow 7 = \mu + z_{0.05}\frac{\sigma}{\sqrt{7}} = \mu - 0.6217\sigma(2)$

$(1) - (2): 1 = 1.903\sigma$

$\sigma = 0.525\,\text{mm}$

$(1): \mu = 8 - 1.282\sigma = 7.33\,\text{mm}$

26

Tip: In questions like this one, it is advisable to designate a new variable for each of the situations. Clearly define new variables and give their distribution, and your working will remain clear.

Let X be the time of a single wait, in minutes.

$X \sim \mathrm{N}(12, 4^2)$

a $\mathrm{P}(X > 20) = 0.0228$

b **i** Total weight for 5 days is $S_5 = \sum_1^5 X_i \sim \mathrm{N}(60, 80)$

$\mathrm{P}(S_5 < 70) = 0.868$

ii Let Y be the number of days her wait is less than ten minutes from a random sample of 5 days.

$\mathrm{P}(X < 10) = 0.3085$

So $Y \sim \mathrm{Bin}(5, 0.3085)$

$\mathrm{P}(Y = 2) = 0.315$

iii Average greater than 10 is equivalent to a total greater than 50.

$\mathrm{P}(S_5 > 50) = 0.868$

c If the total for the first four days is 50 then the average for the week is

$$\frac{50 + X}{5} = 10 + 0.2X$$

$10 + 0.2X \sim \mathrm{N}(12.4, 0.8^2)$

$\mathrm{P}(10 + 0.2\mathrm{X} > 12) = 0.691$

d An average of 14 is equivalent to a total of 70 and an average of 15 is equivalent to a total of 75.

$$\mathrm{P}(\bar{X} < 75 | \bar{X} > 70) = \frac{\mathrm{P}(70 < \bar{X} < 75)}{\mathrm{P}(\bar{X} > 70)} = 0.645$$

27 Let X be the time, in minutes, for Johannes to answer a multiple choice question.

$X \sim \mathrm{N}(1.5, 0.6^2)$

a Let $Y = \sum_1^{35} X_i \sim \mathrm{N}(35 \times 1.5, 35 \times 0.36) = \mathrm{N}(52.5, 12.6)$

$\mathrm{P}(Y > 60) = 0.0173$

b The Central Limit Theorem allows an approximate normal distribution to be applied to the mean or sum of a sample from a non-normal distribution. In this case, the original distribution is already normal, and a sum of a sample will necessarily be normal, without recourse to the CLT.

28 Let $X \sim \mathrm{N}(15, 4^2)$

Taking a sample of size n, require that $\mathrm{P}(\bar{X} > 16) < 0.05$

$\bar{X} \sim \mathrm{N}\left(15, \frac{4^2}{n}\right)$

$\mathrm{P}(\bar{X} > 16) = \mathrm{P}\left(Z > \frac{16 - 15}{4/\sqrt{n}}\right) < 0.05$, where Z is the standard normal.

$\mathrm{P}\left(Z > \frac{\sqrt{n}}{4}\right) < 0.05$

$\frac{\sqrt{n}}{4} > z_{0.95} = 1.645$

$\sqrt{n} > 6.58$

$n > 43.3$

The least n such that the sample mean has this property is $n = 44$.

Exercise 8C

20 Let X be the number of shooting stars seen in one hour. $X \sim \mathrm{Po}(12)$

$\mathrm{P}(X > 20) = 0.0116$

21 Let X_n be the number of white blood cells in n HPF. $X \sim \mathrm{Po}(4n)$

a $\mathrm{P}(X_1 = 7) = 0.0595$

b $\mathrm{P}(X_6 = 28) = 00548$

22 Let X_n be the number of flaws in n metres of wire. $X \sim \mathrm{Po}(1.8n)$

a $\mathrm{P}(X_1 = 1) = 0.298$

b $\mathrm{P}(X_2 \geq 1) = 0.973$

23 a Let X be the number of messages received in one hour. $X \sim \mathrm{Po}(7.5)$

$\mathrm{P}(X > 4) = 0.868$

b Frequency of messages may vary predictably according to time of day. Also, given communication may be in both directions, a text or email conversation would potentially generate a cluster of messages, so that the distribution would not be Poisson, where each event is independent of previous events.

24 a Let X be the number of accidents in a week. $X \sim \mathrm{Po}(\lambda)$

$\mathrm{P}(X \geq 1) = 1 - \mathrm{e}^{-\lambda} = 0.6$

$\lambda = -\ln 0.4 = 0.916$

b $\mathrm{P}(X > 2) = 0.0656$

25 a Let X be the number of outbreaks in a month. $X \sim \text{Po}(\lambda)$

$$\text{P}(X > 2) = 1 - \left(\text{e}^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)\right) = 0.3$$

Solving on the GDC: $\lambda = 1.91$

Then $\text{P}(X < 2) = 0.430$

b Poisson is not likely to be a good distribution; infection outbreaks are not likely to be independent of each other, since they arise from a contagious process. Also, there may be seasonal variations in the probability of an outbreak, so that the probability is not constant throughout the year.

26 $X \sim \text{Po}(6), Y \sim \text{Po}(60)$,

a i $\text{P}(X = 6) = 0.161$

ii $\text{P}(Y = 60) = 0.0514$

b Let U be the number of minutes in a 10 minute period in exactly 6 calls are received.

$U \sim \text{B}(10,0.161)$

$\text{P}(U \geq 5) = 0.0132$

27 a Let X be the number of eagles observed in one day. $X \sim \text{Po}(1.4)$

$\text{P}(X > 3) = 0.0537$

b $\text{P}(X = 2|X \geq 1) = \frac{\text{P}(X=2)}{\text{P}(X\geq 1)} = 0.321$

28 a Let X be the number of mistakes the teacher makes in a piece of homework. $X \sim \text{Po}(1.6)$

$\text{P}(X \geq 2) = 0.475$

b $\text{P}(X = 0) = 0.202$

$\text{P}(X = 1) = 0.323$

$\text{P}(X = 2) = 0.258$

Since the Poisson distribution is unimodal, the most likely number of errors is 1.

c Let Y be the number of pieces of work out of 12 that have no errors.

$Y \sim \text{B}(12, 0.202)$

$\text{P}(Y > 6) = 0.00413$

29 Let X be the number of requests in a single day. $X \sim \text{Po}(1.3)$

a $\text{P}(X = 0) = 0.273$

b $\text{P}(X > 2) = 0.143$

c The probability of a limousine being in use is $p = 0.5 \times \text{P}(X = 1) + \text{P}(X \geq 2)$

$p = 0.550$

Then the expected number of days in 365 that each will be used is $0.550 \times 365 = 201$

30 Let X be the number of cupcakes sold in one day. $X \sim \mathrm{Po}(21.5)$

a $\left(\mathrm{P}(X > 15)\right)^3 = 0.9073^3 = 0.747$

b Let n be the number produced in a batch.

Let Y be the number of cupcakes sold over three days (assuming independence). $Y \sim \mathrm{Po}(64.5)$

Require n such that $\mathrm{P}(Y > n) < 0.05$

$\mathrm{P}(Y > 70) = 0.224$

$\mathrm{P}(Y > 80) = 0.0264$

$\mathrm{P}(Y > 75) = 0.0879$

$\mathrm{P}(Y > 77) = 0.0561$

$\mathrm{P}(Y > 78) = 0.0441$

So the least such n is $n = 78$.

Exercise 8D

19 a 0.35

b $\mathbf{T}^{50} = \begin{pmatrix} 0.875 & 0.875 \\ 0.125 & 0.125 \end{pmatrix}$ is the long-term probability matrix.

c Long-term, 12.5% of cars are manual.

20 a $\mathbf{T} = \begin{pmatrix} 0.95 & 0.8 \\ 0.05 & 0.2 \end{pmatrix}$

b $\mathbf{T}^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.9413 & 0.938 \\ 0.0586 & 0.062 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.938 \\ 0.062 \end{pmatrix}$

Probability of scoring three shots after a miss is 0.938

c Steady state $\mathbf{s}$ is such that $\mathbf{Ts} = \mathbf{s}$

$$\begin{cases} 0.95s_1 + 0.8s_2 = s_1 (1) \\ 0.05s_1 + 0.2s_2 = s_2 (2) \end{cases}$$

$s_1 = 16s_2$

$$\mathbf{s} = \begin{pmatrix} 16/17 \\ 1/17 \end{pmatrix} = \begin{pmatrix} 0.941 \\ 0.0588 \end{pmatrix}$$

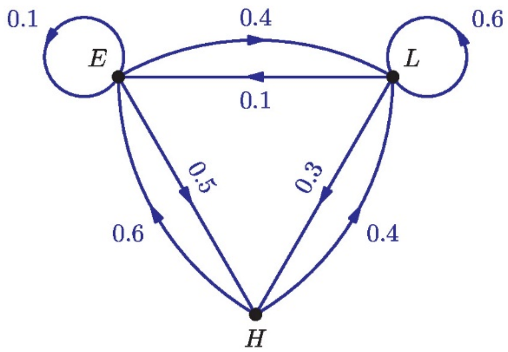
21 a $\mathbf{T} = \begin{pmatrix} 0.8 & 0.2 & 0.1 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.6 \end{pmatrix}$

b $\mathbf{T}^4 \begin{pmatrix} 400 \\ 240 \\ 360 \end{pmatrix} = \begin{pmatrix} 433 \\ 357 \\ 210 \end{pmatrix}$

c Steady state $\mathbf{s}$ is such that $\mathbf{Ts} = \mathbf{s}$ so $(\mathbf{T} - \mathbf{I})\mathbf{s} = 0$

Solving on the GDC: $\mathbf{s} = \begin{pmatrix} 450 \\ 350 \\ 200 \end{pmatrix}$

22 a



b $\mathbf{T}^3\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0.185\\0.496\\0.319\end{pmatrix}$

So the probability of an empty area being heavily populated three years later is 0.319

c $\mathbf{T}^{50}\begin{pmatrix}p\\q\\1-p-q\end{pmatrix}=\begin{pmatrix}0.233\\0.5\\0.267\end{pmatrix}$

The long term probability of being lightly populated is 0.5

23 a For the three states in order being 'bullish', 'bearish' and 'stagnant':

$\mathbf{T}=\begin{pmatrix}0.85 & 0.1 & 0.25\\0.05 & 0.75 & 0.25\\0.1 & 0.15 & 0.5\end{pmatrix}$

b $\mathbf{T}^3\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0.278\\0.513\\0.209\end{pmatrix}$

The probability of a bullish week three weeks after a bearish week is 0.278

c $\mathbf{T}^{50}\begin{pmatrix}p\\q\\1-p-q\end{pmatrix}=\begin{pmatrix}0.515\\0.294\\0.191\end{pmatrix}$

The long term probabilities are 51.5% bullish weeks, 29.4% bearish weeks and 19.1% stagnant weeks.

24 a $\mathbf{T}=\begin{pmatrix}1 & 0.5 & 0 & 0\\0 & 0 & 0.5 & 0\\0 & 0.5 & 0 & 0\\0 & 0 & 0.5 & 1\end{pmatrix}$

b $\mathbf{T}^{50}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}1/3\\0\\0\\2/3\end{pmatrix}$

The long term probability that with a start of \$20 he will leave with \$30 is $\frac{2}{3}$

25 a $\mathbf{T}=\begin{pmatrix}0.7 & 0.4\\0.3 & 0.6\end{pmatrix}$

b $\mathbf{T}^2\begin{pmatrix}0.45\\0.55\end{pmatrix}=\begin{pmatrix}0.5605\\0.4395\end{pmatrix}$

56.05% migrate to A two years later.

c Steady state $\mathbf{s}$ is such that $\mathbf{Ts} = \mathbf{s}$

$$\begin{cases} 0.7s_1 + 0.4s_2 = s_1 (1) \\ 0.3s_1 + 0.6s_2 = s_2 (2) \end{cases}$$

$$s_2 = \frac{3}{4}s_1 \Rightarrow s_1 = \frac{4}{7}, s_2 = \frac{3}{7}$$

The steady state has $\frac{4}{7}$ of the birds migrating to A, and $\frac{3}{7}$ migrating to B.

26 a $\mathbf{T}^2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0.375\\0.5\\0.125\end{pmatrix}$

The probability of a recessive second generation arising from a dominant first generation is 0.125

b $\mathbf{T}^{50}\begin{pmatrix}p\\q\\1-p-q\end{pmatrix} = \begin{pmatrix}0.25\\0.5\\0.25\end{pmatrix}$

The long-term probabilities are 25% dominant, 50% hybrid and 25% recessive

27 a $\mathbf{T} = \begin{pmatrix}0.9 & 0.15\\0.1 & 0.85\end{pmatrix}$

b Eigenvalues λ are such that $\det(\mathbf{T} - \lambda\mathbf{I}) = 0$

$(0.9 - \lambda)(0.85 - \lambda) - 0.015 = 0$

$\lambda^2 - 1.75\lambda + 0.75 = 0$

$(\lambda - 1)(\lambda - 0.75) = 0$

$\lambda = 1$ or 0.75

$\lambda = 1$:

$\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \Rightarrow 0.15y = 0.1x \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$

$\lambda = 0.75$:

$\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix} = \frac{3}{4}\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow y = -x \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$

The eigenvalues are $\lambda_1 = 1$, with eigenvector $v_1 = \begin{pmatrix}3\\2\end{pmatrix}$ and $\lambda_2 = \frac{3}{4}$, with eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$

c $\mathbf{P} = \begin{pmatrix}3 & 1\\2 & -1\end{pmatrix}, \mathbf{D} = \begin{pmatrix}1 & 0\\0 & \frac{3}{4}\end{pmatrix}$

d Proportion after n years is $\boldsymbol{p}_n$ where

$$\boldsymbol{p}_n = \mathbf{PD}^n\mathbf{P}^{-1}\begin{pmatrix}0.65\\0.35\end{pmatrix}$$
$$= \begin{pmatrix}3 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & \left(\frac{3}{4}\right)^n\end{pmatrix}\begin{pmatrix}3 & 1\\2 & -1\end{pmatrix}^{-1}\begin{pmatrix}0.65\\0.35\end{pmatrix}$$
$$= \begin{pmatrix}3 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & \left(\frac{3}{4}\right)^n\end{pmatrix}\left(-\frac{1}{5}\right)\begin{pmatrix}-1 & -1\\-2 & 3\end{pmatrix}\begin{pmatrix}0.65\\0.35\end{pmatrix}$$
$$= \begin{pmatrix}3 & \left(\frac{3}{4}\right)^n\\2 & -\left(\frac{3}{4}\right)^n\end{pmatrix}\begin{pmatrix}0.2\\0.05\end{pmatrix}$$
$$= \begin{pmatrix}0.6 + 0.05 \times \left(\frac{3}{4}\right)^n\\0.4 - 0.05 \times \left(\frac{3}{4}\right)^n\end{pmatrix}$$

The proportion in urban areas after n years is $0.6 + 0.05 \times \left(\frac{3}{4}\right)^n$

e As $n \to \infty$, that proportion tends to 0.6.

28 a

$$\mathbf{T} = \begin{pmatrix}0.8 & 0.6\\0.2 & 0.4\end{pmatrix}$$

b Eigenvalues λ are such that $\det(\mathbf{T} - \lambda\mathbf{I}) = 0$

$(0.8 - \lambda)(0.4 - \lambda) - 0.12 = 0$

$\lambda^2 - 1.2\lambda + 0.2 = 0$

$(\lambda - 1)(\lambda - 0.2) = 0$

$\lambda = 1$ or 0.2

$\lambda = 1$:

$$\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \Rightarrow 0.3y = 0.1x \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

$\lambda = 0.2$:

$$\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{5}\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow y = -x \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

The eigenvalues are $\lambda_1 = 1$, with eigenvector $v_1 = \begin{pmatrix}3\\1\end{pmatrix}$ and $\lambda_2 = \frac{1}{5}$, with eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$

c $\mathbf{P} = \begin{pmatrix}3 & 1\\1 & -1\end{pmatrix}, \mathbf{D} = \begin{pmatrix}1 & 0\\0 & \frac{1}{5}\end{pmatrix}$

d Proportion after n years is $\boldsymbol{p}_n$ where

$$\begin{aligned}
\boldsymbol{p}_n &= \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}\begin{pmatrix}1\\0\end{pmatrix}\\
&= \begin{pmatrix}3 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & \left(\frac{1}{5}\right)^n\end{pmatrix}\begin{pmatrix}3 & 1\\1 & -1\end{pmatrix}^{-1}\begin{pmatrix}1\\0\end{pmatrix}\\
&= \begin{pmatrix}3 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & \left(\frac{1}{5}\right)^n\end{pmatrix}\left(-\frac{1}{4}\right)\begin{pmatrix}-1 & -1\\-1 & 3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= \begin{pmatrix}3 & \frac{1}{5^n}\\1 & -\frac{1}{5^n}\end{pmatrix}\begin{pmatrix}0.25\\0.25\end{pmatrix}\\
&= \begin{pmatrix}0.75 + 0.25 \times \frac{1}{5^n}\\0.25 - 0.25 \times \frac{1}{5^n}\end{pmatrix}
\end{aligned}$$

The probability of the field being diseased after n years is $\frac{1}{4} - \frac{1}{4}\left(\frac{1}{5}\right)^n$

e As $n \to \infty$, that proportion tends to $\frac{1}{4}$.

Mixed Exercise 8

1 **a**

$$\begin{aligned}
\mathrm{E}(X) &= \sum_x x \times \mathrm{P}(X = x)\\
&= (1 \times 0.1) + (2 \times 0.2) + (3 \times 0.3) + (4 \times 0.2) + (5 \times 0.2)\\
&= 3.2
\end{aligned}$$

b $\mathrm{E}(Y) = \mathrm{E}(2 - 3X) = 2 - 3\mathrm{E}(X) = -7.6$

c $\mathrm{Var}(Y) = \mathrm{Var}(2 - 3X) = 9\mathrm{Var}(X) = 14.04$

2 **a**

$$\begin{aligned}
1 &= \sum_x \mathrm{P}(X = x)\\
&= 0.4 + 6p
\end{aligned}$$

$p = 0.1$

b

$$\begin{aligned}
\mathrm{E}(V) &= \sum_v v \times \mathrm{P}(V = v)\\
&= (2 \times 0.4) + (3 \times 0.1) + (5 \times 0.2) + (7 \times 0.3)\\
&= 4.2
\end{aligned}$$

c $\mathrm{E}(W) = \mathrm{E}(9 - 2V) = 9 - 2\mathrm{E}(V) = 0.6$

d $\mathrm{SD}(W) = \mathrm{SD}(9 - 2V) = 2\mathrm{SD}(V) = 4.28$

3 **a**

x	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{1}{13}$	$\frac{5}{26}$	$\frac{4}{13}$	$\frac{11}{26}$

b

$$\begin{aligned}\mathrm{E}(X) &= \sum_x x \times \mathrm{P}(X = x)\\ &= \left(1 \times \frac{2}{26}\right) + \left(2 \times \frac{5}{26}\right) + \left(3 \times \frac{8}{26}\right) + \left(4 \times \frac{11}{26}\right)\\ &= \frac{40}{13}\end{aligned}$$

c

$$\begin{aligned}\mathrm{E}(X^2) &= \sum_x x^2 \times \mathrm{P}(X = x)\\ &= \left(1 \times \frac{2}{26}\right) + \left(4 \times \frac{5}{26}\right) + \left(9 \times \frac{8}{26}\right) + \left(16 \times \frac{11}{26}\right)\\ &= \frac{135}{13}\end{aligned}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \left(\mathrm{E}(X)\right)^2 = \frac{155}{169} \approx 0.917$$

$$\mathrm{Var}(20 - 5X) = 25\mathrm{Var}(X) = 22.9$$

4 **a** Let A be the mass of a bottle and B the mass of an empty case. Let D be the mass of a full case.

$$D = \sum_{i=i}^{5} A_i + B$$

$$\mathrm{E}(D) = 5\mathrm{E}(A) + \mathrm{E}(B) = 6.4 \text{ kg}$$

b $\mathrm{Var}(D) = 5\mathrm{Var}(A) + \mathrm{Var}(B) = 0.52 \text{ kg}^2$

c It is necessary to assume that the mass of each bottle in the case is independent of the mass of each of the others, and of the case.

5 **a** Let X be the height of a tree in metres and $n = 35$.

$$\mathrm{E}(\bar{X}) = \mathrm{E}(X) = 20 \text{ m}$$

$$\mathrm{Var}(\bar{X}) = \frac{1}{n}\mathrm{Var}(X) = 1.57 \text{ m}^2$$

b By the CLT, $\bar{X}$ is approximately normal.

$$\mathrm{P}(\bar{X} < 18) = 0.0553$$

6 Let X be the number of cars and Y the number of motorbikes arriving in a five minute interval.

$X \sim \mathrm{Po}(8)$ and $Y \sim \mathrm{Po}(1.4)$ so $(X + Y) \sim \mathrm{Po}(9.4)$

$\mathrm{P}(X + Y = 10) = 0.123$

7 Let X be the number of tweets in a single day.

a

$$\mathrm{E}\left(\sum_{i=1}^{5} X_i\right) = 5\mathrm{E}(X) = 45$$

$$\mathrm{SD}\left(\sum_{i=1}^{5} X_i\right) = \sqrt{5}\mathrm{SD}(X) = 6.71$$

b It is assumed that the number of tweets on each day is independent of the number on any other day in the same week.

8 a There must be a constant mean rate of events and the events must occur independently of each other.

b $\mathrm{P}(4 < X \leq 10) = \mathrm{P}(X \leq 10) - \mathrm{P}(X \leq 4) = 0.731$

9 a The average rate must be constant and events must be independent of each other. However, it would be reasonable to suppose that arrival rates would vary according to time of day, and that bird behaviour is affected by the behaviour of others – either in terms of flocking or competition avoidance; either way the arrival of one bird would not be independent of the arrival of others.

b $X \sim \mathrm{Po}(12)$

$\mathrm{P}(X > 15) = 0.156$

10 a 0.04

b $\mathbf{T}^3 \begin{pmatrix} 0.42 \\ 0.38 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.246 \\ 0.426 \\ 0.328 \end{pmatrix}$

In three years' time, the market share will be:

Pacey Play 24.6%, Rapid Rate 42.6% and Super Speedy 32.8%

c Steady state $\mathbf{s}$ is such that $\mathbf{Ts} = \mathbf{s}$ so $(\mathbf{T} - \mathbf{I})\mathbf{s} = 0$

Solving on the GDC: $\mathbf{s} = \begin{pmatrix} 0.162 \\ 0.337 \\ 0.501 \end{pmatrix}$

Pacey Play 16.2%, Rapid Rate 33.7% and Super Speedy 50.1%

11 a i $\mathrm{E}(X_1 + X_2) = 2\mathrm{E}(X) = 2\mu$

$\mathrm{Var}(X_1 + X_2) = 2\mathrm{Var}(X) = 2\sigma^2$

ii $\mathrm{E}(3X_1) = 3\mathrm{E}(X) = 3\mu$

$\mathrm{Var}(3X_1) = 9\mathrm{Var}(X) = 9\sigma^2$

iii $\mathrm{E}(X_1 + X_2 - X_3) = \mathrm{E}(X) = \mu$

$\mathrm{Var}(X_1 + X_2 - X_3) = 3\mathrm{Var}(X) = 3\sigma^2$

iv $\mathrm{E}(\bar{X}) = \mathrm{E}(X) = \mu$

$\mathrm{Var}(\bar{X}) = \frac{1}{n}\mathrm{Var}(X) = \frac{\sigma^2}{n}$

12 Let X be the mass of a man in the office block.

$$\sum_{i=1}^{7} X_i \sim \mathrm{N}(7 \times 78, 7 \times 8^2)$$

$$\mathrm{P}\left(\sum_{i=1}^{7} X_i > 600\right) = 0.00537$$

13 $Y = \sum_{i=1}^{6} X_i$

$Y \sim \mathrm{N}(331 \times 6, 2^2 \times 6)$

a $\mathrm{P}(Y < 1980) = 0.110$

b $\mathrm{E}(Y - 6X) = \mathrm{E}(Y) - 6\mathrm{E}(X) = 0$

$\mathrm{Var}(Y - 6X) = \mathrm{Var}(Y) + 36\mathrm{Var}(X) = 24 + 144 = 168$

c $\mathrm{P}(Y - 6X) > 5 = 0.350$

14 Let X be the error in the length of a piece of rope.

By the CLT, $\bar{X}$ has approximately normal distribution for $n = 35$.

$$\bar{X} \sim \mathrm{N}\left(0, \frac{0.5^2}{35}\right)$$

$\mathrm{P}(\bar{X} < 0.1) = 0.882$

15 a If calls have a constant average rate throughout the day and arrive independently of each other then a Poisson distribution would be appropriate. If X is the number of calls in a day then $X \sim \mathrm{Po}(45)$ would be an appropriate model.

b $\mathrm{P}(X > 50) = 0.204$

c $\mathrm{P}(X > 45) = 0.460$

The probability of this event every day for 5 days is $0.460^5 = 0.0207$

16 Let X_n be the number of rainstorms in n weeks. $X_n \sim \mathrm{Po}(2n)$

a $\mathrm{P}(X_4 \geq 8) = 0.547$

b $\mathrm{P}(X_n \geq 1) = 1 - \mathrm{e}^{-2n} > 0.99$

$\mathrm{e}^{-2n} < 0.01$

$$n > \frac{1}{2}\ln 100 = 2.3$$

The least such n is $n = 3$

17 Let X_n be the number of eruptions in n hours. $X_n \sim \mathrm{Po}\left(\frac{20n}{24}\right)$

a $\mathrm{P}(X_1 = 1) = 0.362$

b $\mathrm{P}(X_{24} > 22) = 0.279$

c $\mathrm{P}(X_{0.5} = 0) = 0.659$

d For the first eruption to occur between 3am and 4am requires no eruption for 3 hours followed by at least one eruption in the next hour. Let this event be A.

$$\begin{aligned}\mathrm{P}(A) &= \mathrm{P}(X_3 = 0) \times \mathrm{P}(X_1 > 0)\\ &= 0.0820 \times 0.565\\ &= 0.0464\end{aligned}$$

e Let Y be the number of hours in 8 during which there is at least one eruption.

$Y \sim \mathrm{B}(8, 0.565)$

$\mathrm{P}(Y \geq 6) = 0.247$

f

$$\begin{aligned}\mathrm{P}(X_1 = 1 | X_1 \geq 1) &= \frac{\mathrm{P}(X_1 = 1)}{\mathrm{P}(X \geq 1)}\\ &= \frac{0.362}{0.565}\\ &= 0.641\end{aligned}$$

18 Let X_n be the number of patients arriving at the ER in an hour.

$X_n \sim \mathrm{Po}(14n)$

a $\mathrm{P}(X_{0.25} = 4) = 0.189$

b

$$\begin{aligned}\mathrm{P}(X_1 < 15 | X_1 > 12) &= \frac{\mathrm{P}(12 < X_1 < 15)}{\mathrm{P}(X_1 < 15)}\\ &= \frac{0.212}{0.570}\\ &= 0.372\end{aligned}$$

c Let Y be the number of hours in a sample of 10 hours for which more than 15 patients arrive at the ER.

$\mathrm{P}(X_1 > 15) = 0.331$

$Y \sim \mathrm{B}(10, 0.331)$

$\mathrm{P}(Y \geq 5) = 0.208$

19 a Taking the states in the order R, F, S:

$$\mathbf{T} = \begin{pmatrix} 0.6 & 0.4 & 0.5 \\ 0.1 & 0.4 & 0.2 \\ 0.3 & 0.2 & 0.3 \end{pmatrix}$$

b

$$\mathbf{T}^3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.526 \\ 0.198 \\ 0.276 \end{pmatrix}$$

The probability of a rise three days after a fall is 0.526

c i Steady state $\mathbf{s} = \begin{pmatrix} r \\ f \\ s \end{pmatrix}$ satisfies $\mathbf{Ts} = \mathbf{s}$ and the elements of $\mathbf{s}$ sum to probability 1.

$$\begin{cases} 0.6r + 0.4f + 0.5s = r \\ 0.1r + 0.4f + 0.2s = f \\ 0.3r + 0.2f + 0.3s = s \\ \quad r + \quad f + \quad s = 1 \end{cases}$$

ii

$$\mathbf{s} = \frac{1}{71}\begin{pmatrix} 38 \\ 13 \\ 20 \end{pmatrix}$$

Steady state probabilities are

$\mathrm{P(rise)} = \frac{38}{71}, \mathrm{P(fall)} = \frac{13}{71}, \mathrm{P(same)} = \frac{20}{71}$

20 Let L and S be the quantity of oil in a large and small can respectively.

$L \sim \mathrm{N}(5000, 40), S \sim \mathrm{N}(1000, 25)$

a $\mathrm{P}(L \geq 4995) = 0.785$

b $L - 5S \sim \mathrm{N}(0, 40 + 25 \times 25)$

$\mathrm{P}(L - 5S > 30) = 0.122$

c $L - \sum_{i=1}^{5} S_i \sim \mathrm{N}(0, 40 + 5 \times 25)$

$$\mathrm{P}\left(L - \sum_{i=1}^{5} S_i > 30\right) = 0.00976$$

21 Let X_n be the number of cats visiting Helena's garden over n weeks. $X_n \sim \mathrm{Po}(0.6n)$

i $\mathrm{P}(X_1 = 0) = 0.549$

ii $\mathrm{P}(X_1 \geq 3) = 0.0231$

iii $\mathrm{P}(X_4 \leq 5) = 0.964$

iv Let Y be the number of weeks in a 12 week period in which at least one cat visits Helena's garden

$\mathrm{P}(X_1 \geq 1) = 1 - 0.549 = 0.451$

$Y \sim \mathrm{B}(12, 0.451)$

$\mathrm{P}(Y = 4) = 0.169$

22 $X \sim \mathrm{Po}(8)$

a i $\mathrm{P}(X = 6) = 0.122$

ii
$$\begin{aligned} \mathrm{P}(X = 6 | 5 \leq x \leq 8) &= \frac{\mathrm{P}(X=6)}{\mathrm{P}(5 \leq X \leq 8)} \\ &= \frac{0.122}{0.493} \\ &= 0.248 \end{aligned}$$

b i $\mathrm{E}(\bar{X}) = \mathrm{E}(X) = 8$

$\mathrm{Var}(\bar{X}) = \frac{1}{n}\mathrm{Var}(X) = \frac{8}{n}$

ii A Poisson distribution has mean and variance of the same value.
Since $\mathrm{E}(\bar{X}) = n\mathrm{Var}(\bar{X})$, $\bar{X}$ cannot follow a Poisson distribution for $n > 1$.

c By the CLT, $\bar{X}$ follows an approximately normal distribution for large n.

$\bar{X} \sim \mathrm{N}\left(8, \frac{1}{5}\right)$

i $\mathrm{P}(7.1 < \bar{X} < 8.5) = 0.846$

ii $\mathrm{P}(\bar{X} - 8 \leq k) = 0.95$

$\frac{k}{1/\sqrt{5}} = z_{0.95} = 1.645$

$k = \frac{1.645}{\sqrt{5}} = 0.736$

23 Let X be the length of a fish of this type.

The standard deviation of the mean will be $\frac{4.6}{\sqrt{n}}$

Require $\frac{4.6}{\sqrt{n}} < 0.5$

$n > 9.2^2 = 84.6$

The biologist will need to collect at least 85 fish.

The CLT is not needed for this assessment; at no point does the sample mean need to be assumed normal. It is important that the fish can be assumed to have independent lengths.

24 By the CLT, the total mass M g will have an approximately normal distribution.

$M \sim \mathrm{N}(498 \times 50, 50\sigma^2) = \mathrm{N}(24900, 50\sigma^2)$

$\mathrm{P}(M < 25000) = 0.9577$

$$\frac{25000 - 24900}{\sigma\sqrt{50}} = z_{0.9577} = 1.725$$

$100 = 1.725\sigma\sqrt{50}$

$$\sigma = \frac{100}{1.725\sqrt{50}} = 8.2 \text{ g}$$

25 Let A be the mark of a student in maths and B that student's result in English.

$A \sim \mathrm{N}(63, 64), B \sim \mathrm{N}(61, 71)$

a $A - B \sim \mathrm{N}(2, 133)$

$\mathrm{P}(A - B) < 0 = 0.431$

b For $n = 12, \bar{A} \sim \mathrm{N}\left(63, \frac{64}{12}\right)$ and $\bar{B} \sim \mathrm{N}\left(61, \frac{71}{12}\right)$

$\bar{A} - \bar{B} \sim \mathrm{N}\left(2, \frac{133}{12}\right)$

$\mathrm{P}(\bar{A} - \bar{B} < 0) = 0.274$

Worked solutions

26 Let X_n be the number of worms in a square metre. $X_n \sim \mathrm{Po}(1.2n)$

a $\mathrm{P}(X_2 = 2) = 0.261$

b $[\mathrm{P}(X_1 = 1)]^2 = 0.361^2 = 0.131$

c $\mathrm{P}(X_1 = 0) = 0.301$

If her observations are Y then the probabilities $\mathrm{P}(Y = y)$ for $y \geq 1$ have the same ratios as the equivalent probabilities $\mathrm{P}(X = y)$ but must be scaled up so that they sum to 1.

$$\mathrm{P}(Y = y) = \begin{cases} 0 (y = 0) \\ \dfrac{\mathrm{P}(X = y)}{1 - \mathrm{P}(X = 0)} (y > 0) \end{cases}$$

$$\begin{aligned} \mathrm{E}(Y) &= \sum_{y \geq 0} y \times \mathrm{P}(Y = y) \\ &= \sum_{y \geq 0} y \times \frac{\mathrm{P}(Y = x)}{1 - \mathrm{P}(X = 0)} \\ &= \frac{\mathrm{E}(X)}{1 - \mathrm{P}(X = 0)} \\ &= \frac{1.2}{0.699} \\ &= 1.72 \end{aligned}$$

27 Let X be the number or pairs of sunglasses bought in a week. $X \sim \mathrm{Po}(42.5)$

a $\mathrm{P}(X > 50) = 0.112$

b $\mathrm{P}(X = 41) = 0.0605$

$\mathrm{P}(X = 42) = 0.0613$

$\mathrm{P}(X = 43) = 0.0605$

As a Poisson distribution is unimodal, the most likely number of sunglasses bought is 42.

Comment: Given your knowledge of the probabilities from a Poisson distribution, prove that the mode m of a Poisson $\mathrm{Po}(\lambda)$ is given by

$$m = \begin{cases} \lambda, \lambda + 1 (\lambda \in \mathbb{Z}^+) \\ \lfloor \lambda \rfloor \ (\lambda \notin \mathbb{Z}) \end{cases}$$

where $\lfloor \lambda \rfloor$ is floor of lambda, rounding down to the nearest integer below.

c Require n such that $\mathrm{P}(X \leq n) \geq 0.99$

$\mathrm{P}(X \leq 57) = 0.9807$

$\mathrm{P}(X \leq 58) = 0.9905$

The least such n is 58.

28 a Listing the states as 'flower' and 'grow':

$$\mathbf{T} = \begin{pmatrix} 0.1 & 0.6 \\ 0.9 & 0.4 \end{pmatrix}$$

b The product of eigenvalues is the matrix determinant. and one of the eigenvalues of a transition matrix must be 1.

det $\mathbf{T} = -0.5$, so the eigenvalues are 1 and -0.5

$\lambda = 1$:

$$\begin{cases} 0.1f + 0.6g = f \\ 0.9f + 0.4g = g \end{cases}$$

$3f = 2g$

Eigenvector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda = -0.5$:

$$\begin{cases} 0.1f + 0.6g = -0.5f \\ 0.9f + 0.4g = -0.5g \end{cases}$$

$f = -g$

Eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -0.5 \end{pmatrix}$

d $\mathbf{T}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{PD}^n\mathbf{P}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -0.5 \end{pmatrix}^n \left(-\frac{1}{5}\right) \begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & (-0.5)^n \\ 3 & -(-0.5)^n \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.6 \end{pmatrix}$$
$$= \begin{pmatrix} 0.4 + 0.6(-0.5)^n \\ 0.6 - 0.6(-0.5)^n \end{pmatrix}$$

The probability of flowering in year n is $0.4 + 0.6(-0.5)^n$

e As $n \to \infty$, this probability tends to 0.4.

Worked solutions

9 Statistics

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 9A

7 Use $s_{n-1} = \sqrt{\frac{n}{n+1}}\, s_n$

$$s_{n-1} = \sqrt{\frac{10}{9}} \times 12.4 = 13.1$$

(As a quick check, s_{n-1} should be larger than s_n because samples tend to under-estimate spread.)

8 The question says that $s_{n-1} = 1.118\, s_n$

so $\sqrt{\frac{n}{n-1}} = 1.118$

$$\frac{n}{n-1} = 1.118^2 = 1.249924$$

$$n = 1.249924(n-1)$$

$$0.249924n = 1.249924$$

$$n = 5$$

9 For the first two parts, you can get the answer directly from your calculator, without having to write down any working.

a $\bar{x} = 4.5$

b $s_{n-1}^2 = 3.5$

c The estimate above assume that we have taken a random sample of the population. Claudia's sample may not be representative of the whole population.

d Reliability is about whether a different sample would give similar results. One method to check reliability is test-retest. In Claudia's example, she could take another sample of six snakes and then use the t-test to test whether the two sample means are significantly different.

10 a An outlier should be removed if we believe that it is an error rather than a genuine extreme observation. In this case, the figure of 3498 is very unlikely to be genuine; it is more likely to be a recording error, or someone not taking the survey seriously.

b From the remaining five numbers, using the GDC: $\bar{x} = 2, s_{n-1}^2 = 1.5$

c Content validity of a question is about whether the answers really tell us what we want to know. In this case, if different people are interpreting the question differently, then we don't know how to interpret their answers. For example, different people may interpret 'criminal activity' differently, not be clear whether 'in the last year' means in the last calendar year or in the past 12 months, or not realise that they have been indirectly affected by criminal activity. (Just one of these reasons is sufficient in your answer. When commenting on validity, you should look for reasons why the question may not be valid, rather than simply stating that it is or is not valid.)

d The sample is not random (as it was self-selecting). This may introduce bias.

e People visiting a police station may be more likely to have been affected by crime, so it is likely to be an overestimate.

f Reliability is about how much the results vary between different samples. This could be improved by taking a larger sample size.

11 a Reliability is when similar conclusions are be reached on each occasion the test is conducted in similar circumstances. (You could also say that different samples will lead to similar results or conclusions.)

b This is the parallel forms test for reliability.

c She could use test-retest: This involves giving the same test again (some time later) and comparing the results.

12 a Validity is the extent to which you are measuring what you really want to measure.

(You should learn this definition.)

b Content validity tells us whether the answer to the question will tells us about the thing we want to measure. in this case, an increase in salary may not be the only possible measure of business success, and people's opinions are not the same as actual success.

c One threat to validity is people not answering truthfully; in this case, they may want to be polite to the coach. This could be avoided by using an anonymous questionnaire.

d This is criterion validity, which is about the extent to which the answers to the question correlated with another measure of the same or similar thing.

Exercise 9B

10 Hypotheses:

H_0: The ratio of short plants to tall plants is 3 : 1

H_1: The ratio is not $3 : 1$

Expected frequencies: The total number of plants is 1064, so the expected frequencies are

$$\frac{3}{4} \text{ of } 1064 = 798 \quad \text{and} \quad \frac{1}{4} \text{ of } 1064 = 266$$

There are two groups, so the number of degrees of freedom is 1.

Using the GDC gives $\chi^2 = 0.607,\ p = 0.436$

Since $p > 0.05$, there is no evidence to reject Mendel's prediction.

11 **a** $H_0: N(12, 2.5^2)$ is a good model.

$H_1: N(12, 2.5^2)$ is not a good model.

b Find the probabilities using $N(12, 2.5^2)$ and multiply them by 40:

Probability	0.115	0.230	0.311	0.230	0.115
Expected frequency	4.60	9.18	12.43	9.18	4.60

c The expected frequency of the first group is smaller than 5, so it needs to be combined with the second group; and similarly for the last two groups.

d There are 3 groups left, so the number of degrees of freedom is $3 - 1 = 2$.

e The new observed and expected frequencies are:

Expected	13.78	12.43	13.78
Observed	14	15	11

From the GDC: $\chi^2 = 1.10,\ p = 0.578$

Since $p > 0.10$, there is no evidence that $N(12, 2.5^2)$ is not a good model.

12 **a** H_0: Diet choices are independent of age

H_1: Diet choices are dependent on age.

b The expected values are:

	Vegetarian	**Vegan**	**Eats meat**
11–13	12.77	14.78	14.45
14–15	13.68	15.84	15.48
16–17	6.69	7.74	7.57
17–18	4.86	5.63	5.50

When the expected value for each cell is calculated based on H_0, the value in the lower left cell is less than 5. The table needs to be collapsed in order to get expected values at least 5 in each cell. One option would be to merge the 'vegetarian' column with one of the others, but that would lose much of the detail of the data, and it makes more sense to merge the lower two rows.

c Merging rows, the new expected values are:

	Vegetarian	**Vegan**	**Eats meat**
11–13	12.77	*	14.45
14–15	13.68	15.84	15.48
16–18	11.55	13.38	13.07

Degrees of freedom: $(3 - 1) \times (3 - 1) = 4$

From the GDC: $\chi^2 = 9.57,\ p = 0.0484$

Since $p < 0.05$, there is sufficient evidence to reject H_0 and conclude that diet choice is not independent of age at the school.

13 **a** H_0: $Po(3.5)$ is a good model

H_1: $Po(3.5)$ is not a good model.

b

Number of cars in a minute	0	1	2	3	4	5	≥ 6
Frequency	3	8	10	13	12	9	5
Expected frequency	1.8	6.3	11.1	12.9	11.3	7.9	8.7

Need to combine 0 and 1, so that all expected values are at least 5.

c Merging columns:

Number of cars in a minute	0– 1	2	3	4	5	≥ 6
Frequency	11	10	13	12	9	5
Expected frequency	8.2	11.1	12.9	11.3	7.9	8.5

Degrees of freedom: $6 - 1 = 5$

d From the GDC: $\chi^2 = 2.75,\ p = 0.737$, greater than any reasonable significance level

Do not reject H_0; Po(3.5) is a reasonable model for the data.

14 a H_0: The given model is appropriate.

H_1: The suggested model is not appropriate.

Grade	3	4	5	6	7
Frequency	10	6	4	5	3
Expected frequency	4.5	5.2	5.6	5.6	5.2

Need to combine grades 3 and 4 because the expected frequency for grade 3 is less than 5.

Merging columns:

Grade	3 – 4	5	6	7
Frequency	16	4	5	3
Expected frequency	9.7	5.6	5.6	5.2

Degrees of freedom: $4 - 1 = 3$

From the GDC: $\chi^2 = 5.60,\ p = 0.132 > 0.1$

Do not reject H_0; the limited data is not inconsistent with the proposed model.

15 H_0: The dice is unbiased ($p = \frac{1}{6}$ for all results).

H_1: The dice is biased

Outcome	1	2	3	4	5	6
frequency	6	2	5	7	2	5
Expected frequency	4.5	4.5	4.5	4.5	4.5	4.5

Since all the expected frequencies are less than 5, it would be appropriate to merge columns.

Merging columns

Outcome	1– 2	3– 4	5– 6
frequency	8	12	7
Expected frequency	9	9	9

Degrees of freedom: $3 - 1 = 2$

From the GDC: $\chi^2 = 4.78,\ p = 0.459 > 0.1$

Do not reject H_0; there is insufficient evidence that the dice is biased

16 H_0: Mode of transport is independent of city.

H_1: Mode of transport is not independent of city.

	Amsterdam	Athens	Houston	Jo'burg
Car	12	25	48	24
Bus	18	33	12	18
Bicycle	46	12	7	53
Walk	13	3	0	5

Expected values:

	Amsterdam	Athens	Houston	Jo'burg
Car	29.5	24.2	22.2	33.1
Bus	21.9	18	16.5	24.6
Bicycle	31.9	26.2	24	35.9
Walk	5.7	4.7	4.3	6.4

The expected values for walking are not all at least 5 so this row needs to be merged with another.

It seems sensible to merge with bicycle as the two non-motorised means of transport, in the absence of a strong reason to do otherwise; if the study's focus was on whether or not individuals use their own private means of transportation, it might make more sense to merge 'walk' with 'bus' instead, as the public transport option.

Amended table of expected values:

	Amsterdam	Athens	Houston	Jo'burg
Car	29.5	24.2	22.2	33.1
Bus	21.9	18	16.5	24.6
Bicycle/ Walk	37.6	30.8	28.3	42.2

Degrees of freedom: $(3-1)\times(4-1)=6$

From the GDC: $\chi^2 = 148,\ p \approx 0 \ll 0.05$

Reject H_0; there is sufficient evidence to conclude that mode of transport is not independent of city in this study.

17 a From the data:

$\bar{x} = 25$

$s_{n-1}^2 = 10.8$

x	$]\infty, 10[$	$[10, 20[$	$[20, 30[$	$[30, 40[$	$[40, \infty[$	ν	χ^2
Observed	**200**	**500**	**820**	**500**	**200**		
$\mathbf{N(\mu, \sigma^2)}$	184.1	530.9	790	530.9	184.1	2	7.48
$\mathbf{N(25, \sigma^2)}$	184.1	530.9	790	530.9	184.1	3	7.48
$\mathbf{N(25, 110)}$	169.5	533.8	813.5	533.8	169.5	4	15.31

i Using data to estimate μ and σ, degrees of freedom $n-3=2$

For $\chi_2^2 = 7.48,\ p = 0.0237 < 0.05$: Reject the proposed distribution, it is not consistent with the data.

ii Using data to estimate σ, degrees of freedom $n-2=3$

For $\chi_3^2 = 7.48,\ p = 0.0580 > 0.05$: Do not reject the proposed distribution; it is consistent with the data.

iii Using external assumptions for parameters, degrees of freedom $n - 1 = 2$

For $\chi_4^2 = 15.31$, $p = 0.00410 < 0.05$: Reject the proposed distribution; it is not consistent with the data.

18 a H_0: The data is drawn from a $\mathrm{B}(3, 0.6)$ distribution.

H_1: Otherwise

b

x	0	1	2	3	ν	χ^2
Observed	7	28	95	70		
Expected	12.8	57.6	86.4	43.2	3	35.3

c $\mathrm{P}(\chi_3^2 > 35.3) = 1.07 \times 10^{-7} \ll 0.05$

Reject H_0 at the 5% significance level.
The data are not consistent with the model $\mathrm{B}(3, 0.6)$.

d $\bar{x} = 2.14 = np$

$$p = \frac{214}{300} \approx 0.713$$

e

x	0	1	2	3	ν	χ^2
Observed	7	28	95	70		
$\mathbf{B\left(3, \frac{214}{300}\right)}$	4.71	35.2	87.5	72.6	2	3.31

Need to combine the first two cells since the expected value is less than 5.

x	0,1	2	3	ν	χ^2
Observed	35	95	70		
$\mathbf{B\left(3, \frac{214}{300}\right)}$	39.9	87.5	72.6	1	1.33

Degrees of freedom reduced by one since the data are being used to estimate a parameter.

They are also decreased because the table had to contract.

$\mathrm{P}(\chi_1^2 > 1.33) = 0.249 > 0.05$

Do not reject H_0 at the 5% significance level.

The data are consistent with this model.

19 a $\mathrm{E}(X) = \frac{1}{n}\sum_x x f_x = 2.01$

b H_0: The data is drawn from a $\mathrm{Po}(\lambda)$ distribution.

H_1: The data is not drawn from a $\mathrm{Po}(\lambda)$ distribution.

c

Number of typos	0	1	2	3	4	≥ 5	ν	χ^2
Observed frequency	12	23	29	24	12	0		
Expected frequency	13.4	26.9	27.1	18.1	9.1	5.4	4	9.03

Degrees of freedom reduced by one since the data are being used to estimate parameter λ.

d $\mathrm{P}(\chi_4^2 > 9.03) = 0.0604 > 0.05$

Do not reject H_0 at the 5% significance level.

The data are consistent with the proposed model.

20 a

Midpoint	20.75	22	23	24	25.25
Observed	3	8	14	17	8

$\bar{x} = 23.405$

$s_{n-1}^2 = 1.216$

b

***x* interval in Normal**	$]-\infty, 21.5[$	$[21.5, 22.5[$	$[22.5, 23.5[$	$[23.5, 24.5[$	$[24.5, \infty[$	ν	χ^2
Observed	3	8	14	17	8		
N([23, 1.216]	5.4	11.6	15.9	11.6	5.4	3	6.18

No groups need to be combined, since all expected frequencies are above 5.

c $\nu = 3$

Degrees of freedom reduced by one since the data are being used to estimate parameter σ.

d H_0: The data is drawn from a $\mathrm{N}(23, \sigma^2)$ distribution.

H_1: The data is not drawn from a $\mathrm{N}(23, \sigma^2)$ distribution.

For $\chi_3^2 = 6.18,\ p = 0.103 > 0.05$

Do not reject H_0 at the 5% significance level.

The data are consistent with the proposed model.

21 a

Midpoint	4.75	5.25	5.75	6.5	7.1
Observed	9	18	32	33	8

$\bar{x} = 5.9255$

$s_{n-1}^2 = 0.4313$

Distance	<5	$[5,5.5[$	$[5.5,6[$	$[6,7[$	$[7,\infty[$	ν	χ^2
Observed	9	18	32	33	8		
$\mathbf{N(\ [23,1.216]^2)}$	8.148	17.914	28.410	40.265	5.264	2	3.28

b $\nu = 2$

Degrees of freedom reduced by two since the data are being used to estimate two parameters μ and σ.

c H_0: The data is drawn from a $N(\mu, \sigma^2)$ distribution.

H_1: The data is not drawn from a $N(\mu, \sigma^2)$ distribution.

d For $\chi_2^2 = 3.28,\ p = 0.194 > 0.10$

Do not reject H_0 at the 10% significance level.

The data are consistent with the proposed model.

22 a $\bar{x} = 2.67$

b $s_{n-1}^2 = 1.57$

c i Mean and variance are very dissimilar, which suggests the data are not drawn from a Poisson distribution.

ii H_0: The data is drawn from a $Po(\lambda)$ distribution.

H_1: The data is not drawn from a $Po(\lambda)$ distribution.

X	0	1	2	3	≥ 4	ν	χ^2
Observed	10	20	30	40	50		
$\mathbf{Po(2.67)}$	10.4	27.8	37.1	32.9	41.8	3	6.67

$\nu = 3$

Degrees of freedom reduced by one since the data are being used to estimate parameter λ.

For $\chi_3^2 = 6.67,\ p = 0.083 > 0.05$.

Do not reject H_0 at the 5% significance level.

The data are not inconsistent with the proposed model.

iii H_0: The data is drawn from a $Po(\lambda)$ distribution.

H_1: The data is not drawn from a $Po(\lambda)$ distribution.

X	0	1	2	3	4	>4	ν	χ^2
Observed	10	20	30	40	50	0		
$\mathbf{Po(2.67)}$	10.4	27.8	37.1	32.9	22.0	19.8	4	60.7

$\nu = 3$

Degrees of freedom reduced by one since the data are being used to estimate parameter λ.

For $\chi_4^2 = 60.7,\ p = 2.08 \times 10^{-12} \ll 0.05$

Reject H_0 at the 5% significance level.

The data are not consistent with the proposed model.

d The first assessment is broadly subjective, and does not have an associated significance.

The third method is the most credible because it uses the full information, that there were no results greater than 4.

Exercise 9C

9 The coefficient of determination (R^2) is the square of the Pearson's product-moment correlation coefficient: $R^2 = (-0.879)^2 = 0.773$

10 a The GDC gives the cubic model as $T = -0.00160x^3 + 0.179x^2 + 1.57x - 0.671$

b i Using $x = 39.5$ in the equation gives $T = 242$ years.

ii Although the regression model firs the date perfectly $(R^2 = 1)$, we are extrapolating beyond the range of the data, so the prediction may not be reliable.

(It would also be reasonable to say that the prediction is reliable, because all bodies moving around the Sun have the same relationship between the distance and period, according to Kepler's second law.)

11 a The GDC gives the sinusoidal model as $h = 1.9\sin(0.52t - 0.15) + 6.2$

b The maximum depth is $6.2 + 1.9 = 8.1$ m and minimum depth is $6.2 - 1.9 = 4.3$ m

12 a Choosing linear and power models gives $R^2 = 0.861$ for model A and $R^2 = 0.911$ for model *B*.

b Model B is a better fit (the value of R^2 is larger)

c From the GDC, $D = 89.5p^{-0.785}$

d $D = 89.5 \times 15^{-0.785} = 10.7$ (so around 11)

e This would be extrapolation, so should be treated with caution. (Although the predictions would not be completely unreasonable, as the model predicts that the demand tends to zero as the price increases.)

13 a For each model, we find the predicted values and the squared residuals

$(\text{observed} - \text{predicted})^2$

Model A:

t	observed	prediction	residual squared
1	3.2	3.375	0.031
2	5.8	4.555	1.549
3	7.4	6.149	1.565
4	10.2	8.300	3.609
			6.754

SSres = 6.75

Model B:

t	observed	prediction	residual squared
1	3.2	4.27491	1.155
2	5.8	5.221386	0.335
3	7.4	6.377416	1.046
4	10.2	7.789393	5.811
			8.347

SSres = 8.35

b Model A, because it has the smaller sum of squared residuals.

14 **a** Using a quadratic model on the GDC:

$s_{\text{girls}} = -0.171t^2 + 13.2t + 25.1$

$s_{\text{boys}} = -1.26t^2 + 19.1t + 13.1$

b The models are not based on the same number of data points, so SSres could be different just because different number of values are being added.

c $R^2_{\text{girls}} = 0.915,\ R^2_{\text{boys}} = 0.844$

The girls' model is a better fit

15 **a** From the GDC, $T = 0.2t^2 - 7.13t + 99.8$

b $R^2 = 0.994$

c The high value of R^2 indicates that the model is a good fit in the range $2.5 \le t \le 15$. However, the model predicts that the temperature will increase again for $t > 17.8$, which is clearly not realistic.

16 The comparison may not be valid because the two models have different numbers of parameters to be estimated (a quadratic model has three unknown parameters and a cubic model has three).

Exercise 9D

Note: When finding a confidence interval using the t-distribution, you may get a slightly different answer than given here; this is because different calculators use slightly different approximations for t-values.

13 Using the GDC, $\bar{x} = 4.14$, $s_{n-1} = 0.717$.

Using the t_7 distribution ($n = 8$), the confidence interval is $(3.66, 4.62)$

14 Using the t-interval the GDC with $n = 20$ (t_{19} distribution):

$\bar{x} = 63.8, s_{n-1} = \sqrt{18.6}$

The confidence interval is $62.1 < \mu < 65.5$

15 Using the z-interval with $n = 85, \bar{x} = 806, \sigma = 124$; the confidence interval is $780 < \mu < 832$

16 a From the GDC, $\bar{x} = 26.8$ and the z-interval is $(23.8, 29.8)$

b The population standard deviation is known, so we use the z-interval.

17 a Using the t-interval $n = 80$ (t_{79} distribution), $\bar{x} = 8.6, s_{n-1} = \sqrt{1.5}$:

The confidence interval is $8.32 < \mu < 8.87$.

b It is assumed that the population distribution is normal.

18 a $s_{n-1}^2 = s_n^2 \times \frac{n}{n-1} = 3.8^2 \times \frac{15}{14} = 15.5\ (°\text{C})^2$

b Using t-interval with $n = 15$ (t_{14} distribution), $\bar{x} = 18.6, s_{n-1} = \sqrt{15.5}$:

The confidence interval is $16.8 < \mu < 20.4$.

19 a We need to assume that the population of concentrations is normally

distributed. (This is required in order to use the t-distribution.)

b From the GDC, $\bar{x} = 22, s_{n-1} = 2.88$.

Using the t-interval with $n = 8$ (t_7 distribution); the confidence interval is $(20.1, 23.9)$.

c The confidence interval does not contain 20, suggesting that the student is not preparing the solutions correctly.

20 a The population variance is unknown, so use the t-interval.

We need to assume that the population distribution of the times is normal.

From the GDC, with $n = 10, \bar{x} = 3.2, s_{n-1} = \sqrt{0.6}$:

The confidence interval is $2.75 < \mu < 3.65$.

b The data are consistent with his belief; the confidence interval extends below 3 minutes.

21 a $s_{n-1}^2 = \frac{n}{n-1} s_n^2 = \frac{20}{19} \times 11.56 = 12.2 \text{ hours}^2$

b Using the t-interval with $n = 20, \bar{x} = 26.5, s_{n-1} = \sqrt{12.2}$; the confidence interval is $25.1 < \mu < 27.9$.

The population distribution is assumed to be normal.

c Since 28 is outside the interval in part **b**, the data does not support this claim.

22 a $s_{n-1} = s_n\sqrt{\frac{n}{n-1}} = 4.6 \times \sqrt{\frac{20}{19}} = 4.72$

b Using the t-interval with $\bar{x} = 149, s_{n-1} = 4.72, n = 20$; the confidence interval is $(147, 151)$

c The confidence interval is entirely below 153, so the machine does not need adjusting.

d No, as 150 is inside the confidence interval.

23 a

	$[144.5, 155.5[$	$[155.5, 160.5[$	$[160.5, 165.5[$	$[165.5, 170.5[$	$[170.5, 180.5[$	$[180.5, 190.5[$
x	150	158	163	168	175.5	185.5
f	4	7	12	14	14	9

From the GDC, $\bar{x} = 169$, $s_{n-1} = 9.90$

Using t-interval with $n = 60$, the confidence interval is $(166, 172)$.

b The working above uses the midpoints of intervals, instead of the actual data values, to estimate the mean and standard deviation.

24 a Using the z-interval with $\bar{x} = 17.8$, $\sigma = \sqrt{8.6}, n = 12$, the 80% confidence interval is $(16.7, 18.9)$.

b The entire confidence interval is above 16.5, which suggest that the mean maximum daily temperature is more than 16.5 °C.

c Yes – the new interval is $(16.4, 19.2)$ which goes below 16.5, so this is not sufficient evidence that the mean maximum daily temperature is more than 16.5 °C.

25 a Using z-interval (since the population standard deviation is known) with $\bar{x} = 3.7$, $\sigma = 2.6, n = 80$, the 95% confidence interval is $(3.13, 4.27)$.

b Yes – the sample mean is still approximately normal because the sample size is large (using the central limit theorem).

26 a The difference is calculated as (blood pressure after) – (blood pressure before).

Subject	1	2	3	4	5	6	7	8	9	10	11	12
Before	92	107	121	85	109	135	141	98	112	126	135	108
After	98	121	123	102	132	126	135	98	127	137	122	128
Difference	6	14	2	17	23	−9	−6	0	15	11	−13	20

b Using the final row of the table as the values, the GDC gives $\bar{x} = 6.67, s_{n-1} = 11.9$

The 90% t-interval with $n = 12$ is $(0.507, 12.8)$

c The confidence interval is entirely above zero, which suggests that the mean change is positive, i.e. that there is on average an increase in blood pressure.

Exercise 9E

15 Let $\bar{X}$ be the mean wingspan of 6 sampled butterflies, in cm.

$\bar{X} \sim \mathrm{N}\left(\mu, \frac{1}{6}\right)$

$\mathrm{H}_0: \mu = 10$

$\mathrm{H}_1: \mu > 10$ (two tailed test)

From the GDC, using $N\left(10, \frac{1}{6}\right)$, $p = 0.00164 < 0.05$

Reject the null hypothesis; there is evidence that the mean wingspan of these butterflies is larger. The scientist's suspicions are justified.

16 a $s_{n-1}^2 = \frac{20}{19} \times 12.635 = 13.3$

b The population variance has been estimated from the sample, and the population distribution is normal.

c H_0: $\mu = 65.7$, H_1: $\mu < 65.9$ where μ is the population mean score in Juan's school

From the GDC, using one-tailed t-test with $\bar{x} = 65.9$, $s_{n-1} = \sqrt{13.3}$, $n = 20$

$p = 0.00139 < 0.1$, so there is sufficient evidence to support Juan's belief.

17 a $\mathrm{H}_0: \mu = 7.8$, $\mathrm{H}_1: \mu < 7.8$

b The distribution will be normal, but the standard deviation needs to be divided by $\sqrt{10}$.

$\bar{X} \sim \mathrm{N}\left(7.8, \frac{0.8}{\sqrt{10}}\right)$

c The critical region is $\bar{X} < c$, where $P(\bar{X} < c) = 0.05$

From the GDC, $c = 7.38$, so the critical region is $\bar{X} < 7.38$

18 a The population variance is known, and the population distribution is normal

b H_0: $\mu_2 - \mu_1 = 5$, H_1: $\mu_2 - \mu_1 \neq 5$, where μ_1 and μ_2 are battery lives of KX03 and KX04, respectively.

Most GDCs can only test for $\mu_1 \neq \mu_2$, so we can add 5 to all the value in the first row and the perform the test on the following data:

KX03	43	41	40	31	27	40		
KX04	42	34	27	35	38	31	32	41

We are now testing H_0: $\mu_1 = \mu_2$, $\mathrm{H}_1: \mu_2 \neq \mu_2$ with $\sigma_1 = \sigma_2 = \sqrt{16.8}$, using a 2-sample z-test.

$p = 0.366 > 0.05$, so there is insufficient evidence for the supplier's belief.

19 a Each tree has produced a first and second harvest apple; the data are paired explicitly by source.

b Let D be the difference in harvest (second minus first).

Tree	A	B	C	D	E	F	G	H	I	J
d	−0.5	0.2	−0.5	−0.6	−0.8	0.2	−0.6	−1.2	0.8	−1

$\text{H}_0: \mu_D = 0$

$\text{H}_1: \mu_D < 0$ (one-tailed)

From the GDC using a t-test: $p = 0.0352 < 0.05$

Reject H_0; there is sufficient evidence to support the claim.

20 a

Room	A	B	C	D	E		F	G	H	I	J
d	0.2	0.3	0	0.1	0.4		0.1	0.1	0.3	0.2	0

b The GDC gives the standard deviation for the sample of D to be $s_n = 0.137$

c $\text{H}_0: \mu_D = 0$

$\text{H}_1: \mu_D > 0$ (one-tailed)

From the GDC using a t-test: $p = 0.000151 \ll 0.10$

Reject H_0; there is sufficient evidence to support the belief.

21 $\text{H}_0: \mu = 24,\ \text{H}_1: \mu > 24$

We use a z-test because the population standard deviation is known and the population distribution is normal

From the GDC, 1-sample z-test gives $p = 0.009211 < 0.01$, so there is sufficient evidence that the level is above average.

22 We need to find the critical region for the test $\text{H}_0: \mu = 60,\ \text{H}_1:\ \mu \neq 60$ at the 5% significance level.

The critical region is of the form $\bar{X} < c_1$ or $\bar{X} > c_2$,
where $P(\bar{X} < c_1) = \text{P}(\bar{X} > c_2) = 0.025$.

Using $\bar{X} \sim \text{N}\left(60, \frac{1}{\sqrt{5}}\right)$, we find $c_1 = 59.1, c_2 = 60.9$.

So the critical region is $\bar{X} < 59.1$ or $\bar{X} > 60.9$.

23 a For maths, $s_{n-1}^2 = \frac{8}{7} \times 126 = 144$, For English: $s_{n-1}^2 = \frac{6}{5} \times 102 = 122.4$

b We need to take away 20 from the maths mean and then test whether it is still higher than the English mean.

Do the 2-sample t-test with the hypotheses $\text{H}_0: \mu_1 = \mu_2,\ \text{H}_1: \mu_1 > \mu_2$ and the following data.

	Sample mean	s_{n-1}	n
Maths	115	12	8
English	98	$\sqrt{122.4}$	6

From the GDC, $p = 0.00494 < 0.1$, so there is sufficient evidence that maths homework does take at least 20 minutes longer than the English homework.

c Homework times distributed normally (in order to use the t-test), with equal variances (in order to use the pooled test).

24 a $\mathrm{H}_0: \mu = 800,\ \mathrm{H}_1: \mu > 800$

Use a z-test with $n = 50, \sigma = 100, \bar{x} = 829.4$

$p = 0.0188 < 0.05$, so there is sufficient evidence to support the claim.

b The z-test only requires that the sample mean has a normal distribution (and not the population). Since n is large, the Central Limit Theorem says that the sample mean has an approximately normal distribution, so the test is still valid and the worry is not justified.

25 a The weights before and after need to have a normal distribution with a common variance.

b Using the pooled 2-sample t-test with $\mathrm{H}_1:\ \mu_1 < \mu_2$, we get $p = 0.359$.

c

Mouse	1	2	3	4	5	6
Difference	6	16	0	3	6	1

d The old assumptions are no longer needed. The differences need follow a normal distribution (in the population)

e Fewer assumptions; eliminates differences between individual mice

f Using a t-test with $\mathrm{H}_0: \mu = 0, \mathrm{H}_1: \mu > 0$ gives $p = 0.0368$.

g The paired test is more appropriate because the two measurements come from the same mouse. $0.0368 < 0.1$, so there is evidence that the average weight has increased after taking the drug.

26 a $\mathrm{Var}(D) = \mathrm{Var}(C - T) = \mathrm{Var}(C) + \mathrm{Var}(T) = 8^2 + 5^2 = 89$

b Assuming that the average mark on the two tests is the same,
$\mathrm{E}(D) = 0, \mathrm{E}(C) - \mathrm{E}(T) = 0$

c $\mathrm{H}_0:\ \mu_C = \mu_T,\ \mathrm{H}_1: \mu_C > \mu_T$

Assuming H_0 is true, $D = C - T \sim \mathrm{N}(0, 89)$

Use the data for D: 3, 11, 5, -5, 7, 5, 1, -1

to test $\mathrm{H}_1: \mu_D = 0,\ \mathrm{H}_1: \mu_D > 0$ using a z-test with $\sigma = \sqrt{89}$

From the GDC, $p = 0.165 > 0.1$, insufficient evidence to reject H_0

d There is insufficient evidence to support Johannes's belief.

27 a $\mathrm{H}_0: \mu_D = 0,\ \mathrm{H}_1: \mu_D < 0$

b Assuming H_0 is true, and that the population standard deviation is unchanged,
$\mathrm{E}(D) = 0, \mathrm{Var}(D) = \mathrm{Var}(A) + \mathrm{Var}(B) = 0.86^2 + 0.86^2 = 1.4792$ and

$\mathrm{Var}(\bar{D}) = \dfrac{\mathrm{Var(D)}}{12} = 0.1233$

Hence the distributions are $D \sim \mathrm{N}(0, 1.4792), \bar{D} \sim \mathrm{N}(0, 0.1233)$

c The test is one-tailed, so the critical region is $\bar{D} < c$, where $\mathrm{P}(\bar{D} < c) = 0.05$

Using $\bar{D} \sim \mathrm{N}(0, 0.1233)$, the critical region is $\bar{D} < -0.578$

Note: The calculation for the distributions of D and $\overline{D}$ are only valid if the random variables A and B are independent. Is that likely to be the case here? You may want to consider the extreme case, where the cleaning software decreases all times by the same value – what would the distribution of D be in that case?

Exercise 9F

21 $\mathrm{H_0}: p = 0.78,\ \mathrm{H_1}: p > 0.78$ where p is the population proportion of cases where the treatment is effective.

Assuming $\mathrm{H_0}$ is true , $X \sim \mathrm{B}(60, 0.78)$

$\mathrm{P}(X \geq 52) = 1 - \mathrm{P}(X \leq 51) = 0.0659 > 0.02$

There is insufficient evidence to support the doctor's belief.

22 $\mathrm{H_0}: p = 0.48,\ \mathrm{H_1}: p < 0.48$ where p is the proportion of students with a summer job in Jay's school.

Assuming $\mathrm{H_0}$ is true, $X \sim \mathrm{B}(30, 0.48)$

$\mathrm{P}(X \leq 10) = 0.0761 < 0.1$

There is sufficient evidence that Jay's suspicion is justified.

23 The average of 82 rainy days per year means that $p = \frac{82}{365}$, where p is the proportion of rainy days.

$\mathrm{H_0}: p = \frac{82}{365},\ \mathrm{H_1}: > \frac{82}{365}$

Assuming $\mathrm{H_0}$ is true, $X \sim \mathrm{B}\left(365, \frac{82}{365}\right)$

$\mathrm{P}(X \geq 96) = 1 - \mathrm{P}(X \leq 95) = 0.0471 < 0.05$

There is sufficient evidence that it rains more often on average.

24 a Po(47)

b The number of errors per ten pages is $X \sim \mathrm{Po}(\lambda)$

$\mathrm{H_0}: \lambda = 47,\ \mathrm{H_1}: \lambda < 47$

Note: You can also say $\mathrm{H_0}: \lambda = 4.7,\ \mathrm{H_1}: \lambda < 4.7$ where λ is the average number of errors per page. But you would still use $X =$ number of errors on ten pages, $X \sim \mathrm{Po}\ (10\lambda)$

Assuming $\mathrm{H_0}$ is true, $P(X \leq 35) = 0.0420 < 0.05$

There is sufficient evidence that the average number of errors per page has decreased.

25 a Po(2.4)

b $\mathrm{H_0}: \lambda = 12,\ \mathrm{H_1}: \lambda > 12$ where λ is the average number of flaws per 100 m.

[Or $\lambda = 2.4,\ \lambda > 2.4$ where λ is the average number of flaws per 20 m]

The number of flaws in 20 m is $X \sim \mathrm{Po}(2.4)$.

$\mathrm{P}(X \geq 5) = 1 - \mathrm{P}(X \leq 4) = 0.0959 > 0.05$

There is insufficient evidence that the average number of flaws has increased.

26 a $r = 0.900$

b $H_0: \rho = 0$

$H_1: \rho > 0$ (one tail)

c $p = 0.0185$

d $p < 0.05$ so reject H_0 and conclude that there is significant evidence of positive correlation.

27 $H_0: \rho = 0$

$H_1: \rho > 0$ (one tail)

From the GDC: $r = 0.745, p = 0.0447$

$p < 0.10$ so reject H_0 and conclude that there is significant evidence of positive correlation.

28 a $H_0: \rho = 0, H_1: \rho < 0$

b The sample correlation coefficient is $r = -0.695$, p-value $= 0.0415 < 0.05$

There is sufficient evidence of negative correlation.

29 a If p is the probability of a person voting for party Z.

$H_0: p = 0.38$

$H_1: p > 0.38$

b If $X \sim \text{B}(200, 0.38)$, require least integer c such that $P(X \geq c) < 0.05$.

$\text{P}(X \geq 87) = 0.0639$

$\text{P}(X \geq 88) = 0.0478$

The critical region is $X \geq 88$.

c Since 86 is not in the critical region, cannot reject H_0; there is insufficient evidence that support has increased.

30 $H_0: \rho = 0$

$H_1: \rho \neq 0$

From the GDC (using 2-tail test): $r = -0.0619$, $p = 0.0884 > 0.05$

Cannot reject H_0; there is insufficient evidence of correlation.

31 **a** 0.820 (from the GDC)

b $H_0: \rho = 0$, $H_1: \rho \neq 0$

Two-tail test, $p = 0.0239 > 0.02$

Insufficient evidence of correlation

c For a one-tail test, $p = 0.0120 < 0.02$, so there is sufficient evidence of positive correlation.

32 a Let X be the number of buses arriving in a random 10 minute interval.

If the buses arrive at a constant average rate, independently of each other, then $X \sim \text{Po}(\lambda)$ for some average rate λ.

$H_0: \lambda = 1$

$H_1: \lambda < 1$

b Let X_6 be the number of buses arriving in a random 60 minute interval. $X_6 \sim \text{Po}(6\lambda)$

Require greatest integer c such that under H_0, $\text{P}(\text{X}_6 \leq c) \leq 0.1$

$\text{P}(\text{X}_6 \leq 3) = 0.151$

$\text{P}(\text{X}_6 \leq 2) = 0.0620$

The critical region for the test is $X_6 \leq 2$

c Let X_{30} be the number of buses arriving in a random 60 minute interval. $X_{30} \sim \text{Po}(6\lambda)$

Require greatest integer d such that under H_0, $\text{P}(\text{X}_{30} \leq d) \leq 0.1$

$\text{P}(\text{X}_{30} \leq 23) = 0.115$

$\text{P}(\text{X}_{30} \leq 22) = 0.0806$

The critical region for the test is $X_{30} \leq 22$

33 a $\text{H}_0: \lambda = 6.3, \text{H}_1: \ \lambda > 6.3$, where λ is the average number of floods per year in Gyan's district.

b Assuming H_0 is correct, the number of floods in five years is $X \sim \text{Po}(31.5)$.

The critical region is $X \geq c$, where c is the smallest value for which $P(X \geq c) < 0.05$.

$\text{P}(X \geq 41) = 0.0590 > 0.05$

$\text{P}(X \geq 42) = 0.0421 < 0.05$

The critical region is $X \geq 42$.

34 a Let X be the number of residence in a random sample of 100 who use the post office.

Assuming H_0 is true, $X \sim \text{B}(100,0.1)$.

The post office will be closed if $X \leq c$, where c is the smallest value for which $\text{P}(X \leq c) < 0.05$.

$\text{P}(X \leq 5) = 0.0576 > 0.05$

$\text{P}(X \leq 4) = 0.0237 < 0.05$

So the post office will be closed if 4 or fewer of the 100 sampled residents use it.

For it to remain open, at least 5 residents would need to say that they use it, which is at least 5% of the sample.

b Repeat the same calculation with $X \sim \text{B}(200,0.1)$.

$\text{P}(X \leq 13) = 0.0566 > 0.05$

$\text{P}(X \leq 12) = 0.0320 < 0.05$

For the post office to remain open we need $X \geq 13$, which is 6.5% of the sample.

35 a $\text{H}_0: \lambda = 1.7, \text{H}_0: \lambda < 1.7$, where λ is the mean number of complaints per day

b The test is one-tailed, so the critical region is of the form $X \leq c$.

Using $X \sim \text{Po}(1.7)$, $\quad \text{P}(X = 0) = 0.182 > 0.05$

So there is no c such that $\text{P}(X \leq c) < 0.05$.

The null hypothesis can never be rejected.

c We found above that $\mathrm{P}(X = 0) = 0.182 = 18.2\%$.

We need $\mathrm{P}(X = 0)$ to be less than $n\%$, so the smallest value of n is 19.

d Let Y be the number of complaints during the ten days; then $Y \sim \mathrm{Po}(17)$.

$\mathrm{P}(X \leq 11) = 0.0847 > 0.05$

$\mathrm{P}(X \leq 10) = 0.0261 < 0.05$

The critical region is $X \leq 10$.

e We need $\mathrm{P}(X = 0)$ to be less than 0.01.

$X \sim \mathrm{Po}(1.7d)$ where d is the number of days

d	$\mathbf{P}(X = 0)$
1	$0.182 > 0.01$
2	$0.0334 > 0.01$
3	$0.00610 < 0.01$

The smallest number of days required is 3.

36 a Let p be the probability of a storm happening in a given year, and let X be the number of storms in ten years. Then $X \sim \mathrm{B}(10, p)$. The hypotheses for Greta's test are

$\mathrm{H}_0 : p = 0.01, \ \ \mathrm{H}_1 : p > 0.01$

$\mathrm{P}(X \geq 2) = 0.004\,27 < 0.05$, so there is sufficient evidence that the storms have become more frequent.

b For David's test, $X \sim \mathrm{Po}(10\lambda)$ and the hypotheses are $\mathrm{H}_0 : \lambda = 0.01, \ \ \mathrm{H}_1 : \ \lambda > 0.01$

$\mathrm{P}(X \geq 2) = 0.00468 < 0.05$, so there is sufficient evidence that the storms have become more frequent.

c The significance level needs to be between $0.004\,27$ and $0.004\,68$; for example, 0.45%.

(With this significance level, Greta would conclude that there is evidence of an increase, but David would not.)

Exercise 9G

15 a H_0: defendant is innocent

H_1: defendant is guilty

b Type I error means rejecting the null hypothesis when it is true; so finding a person guilty when they are actually innocent.

c Type II means failing to reject the null hypothesis when it is false; so finding a person innocent when they are actually guilty.

16 a Type II error; customers will be assuming the dice are fair (not rejecting H_0: dice are fair) when in fact they are biased (H_0 is false).

b Let X be the number of times a 6 is rolled. Assume X follows a binomial $\text{B}(15, p)$ for some probability p.

$$\text{H}_0: p = \frac{1}{6}$$

$$\text{H}_0: p < \frac{1}{6}$$

Tip: It could be reasonably argued that we do not need to assume a binomial; you could reframe the test by saying that $\text{H}_0: X \sim \text{B}\left(15, \frac{1}{6}\right)$.

H_1: X follows some other distribution but, in that case, without other considerations, you have to take a two-tailed approach. Since the customer in this question has determined that $X = 0$ is her critical region, it is implicit that you need a one tailed test. It is not unreasonable to imagine that dice rolls will be independent with a constant probability (unless the house has some sort of magnetic apparatus or remote control effect which can affect dice differently from roll to roll), so assuming a binomial at the start is sensible.

c Type I error: H_0 is true but the result lies in the critical region (in this case, no roll of a 6).

Under H_0, $\text{P}(X = 0) = \left(\frac{5}{6}\right)^{15} = 0.0649$

17 a This is the probability of Type I error (concluding that H_0 is false when it is in fact true).

Since the volume is a continuous variable, the probability of Type I error equals the significance level of the test, so it is 0.05.

b We are still looking for the probability of a Type II error, which is the probability of the sample mean falling in the critical region given that the null hypothesis is true.

In this case, $\bar{X} \sim \text{N}\left(2, \frac{0.1^2}{10}\right)$ and so $\text{P}(\bar{X} < 1.95) + \text{P}(\bar{X} > 2.05) = 0.114$

18 a Let X be the number of people attending the gym over the course of an hour. $X \sim \text{Po}(\lambda)$

$\text{H}_0: \lambda = 20$

$\text{H}_1: \lambda > 20$

b If the mean has not increased, $X \sim \text{Po}(20)$

$\text{P}(\text{X} > 28 | \lambda = 2) = 0.0343$

19 a Since length is a continuous variable, the probability of a Type I error equals the significance level of the test, 0.05.

b The probability of a Type II error is the probability of NOT being in the critical region when $\mu = 9.8$.

To find the critical region, we need to find c such that $P(\bar{X} < c) = 0.05$ when $\bar{X} \sim \text{N}\left(10.5, \frac{1.8^2}{6}\right)$

From the GDC, $c = 9.29$

When $\bar{X} \sim \text{N}\left(9.8, \frac{1.8^2}{6}\right)$, $P(\bar{X} > 9.29) = 0.756$

c Increase the sample size.

20 a $H_0: p = 0.8$

$H_1: p < 0.8$

i Type I error: Reject H_0 when it is true.

Daniel concludes that germination rate is less than 80% (fewer than 14 germinate) when the true rate is indeed 80%.

ii Type II error: Fail to reject H_0 when it is false.

Daniel concludes that germination rate is 80% (at least 14 germinate) when the true rate is in fact less than 80%.

b This is the probability of being in the critical region when the null hypothesis is true.

So $X \sim B(20, 0.8)$, $P(X < 14) = P(X \leq 13) = 0.0867$

c This is the probability of being outside the critical region when p is in fact 0.75.

$X \sim B(20, 0.75)$, $P(X \geq 14) = 1 - P(X \leq 13) = 0.786$

d To reduce the chance of Type II error while increasing the chance of Type I error, increase the critical region (increase the limit from 14 to a higher number).

To reduce the chance of both Types of error, use a greater number of seeds.

21 a Let X be the number of beta particles emitted in one second.

$X \sim Po(\lambda)$

$H_0: \lambda = 2$

$H_1: \lambda > 2$

b Let X_5 be the number of beta particles in a five second period. $X_5 \sim Po(5\lambda)$

Significance $= P(\text{Type I error}) = P(X_5 > 15 | \lambda = 2) = 1 - P(X_5 \leq 15) = 0.0487$ (using $X \sim Po(10)$)

c $P(\text{Type II error}) = (X_5 \leq 15 | \lambda = 3) = 0.568$ (using $X \sim Po(15)$)

22 a The probability of a Type I error is determined by the significance level of the test (length is a continuous variable, so the probability exactly equals the significance level), so is not affected by sample size.

b The probability of a Type II error is reduced when the sample size is increased.

23 a $H_0: p = \frac{2}{3}$, $H_1: p > \frac{2}{3}$

b The probability of Type I error is the probability of being in the critical region when H_0 is true. We need to find the critical region whose probability is less than 0.1.

Assuming H_0 is true, the number of bullseyes out of 12 attempts is $X \sim B\left(12, \frac{2}{3}\right)$

$P(X \geq 10) = 1 - P(X \leq 9) = 0.181 > 0.1$

$P(X \geq 11) = 1 - P(X \leq 10) = 0.0540 < 0.1$

The critical region is $X \geq 11$.

When $X \sim B\left(12, \frac{5}{6}\right)$, the probability of a Type II error is $P(X \leq 10) = 0.619$

24 a $H_0: \lambda = 18.4$, $H_1: \lambda < 18.4$

b The critical region for the test is of the form $X \le c$, where $X \sim \text{Po}(18.4)$

The probability of a Type II error is $\text{P}(Y > c)$ where $Y \sim \text{Po}(16.2)$

$\text{P}(Y > 13) = 1 - \text{P}(Y \le 13) = 0.741 > 0.7$

$\text{P}(Y > 14) = 1 - \text{P}(Y \le 14) = 0.651 < 0.7$

so $c = 14$ and the critical region for the test is $X \le 14$.

The probability of a Type I error, using $X \sim \text{Po}(18.4)$, is $\text{P}(X \le 14) = 0.183$

25 a Let X be the number of red ties in the sample.

Assuming H_0 is true (so there are 6 red ties and 10 green ties), the probability of rejecting H_0 is

$\text{P}(X > 1) = \text{P}(2 \text{ red}, 1 \text{ blue}) + \text{P}(3 \text{ red})$

$$= \left(3 \times \frac{6}{16} \times \frac{5}{15} \times \frac{10}{14}\right) + \left(\frac{6}{16} \times \frac{5}{15} \times \frac{4}{14}\right) = \frac{17}{56}$$

(You may want to draw a tree diagram to check this calculation)

b Assuming H_1 is true (so there are 11 red ties and 5 blue ties), The probability of not rejecting H_0 is

$\text{P}(X \le 1) = \text{P}(3 \text{ blue}) + \text{P}(2 \text{ blue}, 1 \text{ red})$

$$= \left(\frac{5}{16} \times \frac{4}{15} \times \frac{3}{14}\right) + \left(3 \times \frac{5}{16} \times \frac{4}{15} \times \frac{11}{14}\right) = \frac{3}{14}$$

Mixed practice 9

1 a $s_{n-1}^2 = \frac{n}{n-1} s_n^2 = 25.3 \text{ cm}^2$

b The best estimate of the standard deviation is $s_{n-1} = \sqrt{25.3}$.

From the GDC, the confidence interval is then $121 < \mu < 126$.

2 a With the correct numbers, the expected values found from the GDC are

	Year 7	**Year 8**	**Year 9**	**Year 10**	**Year 11**
Pizza	5.47	9.70	8.95	11.19	9.70
Hot dog	4.62	8.19	7.56	9.45	8.19
Lasagne	5.96	10.56	9.75	12.18	10.56
Burger	5.96	10.56	9.75	12.18	10.56

Since the *expected* value of Year 7 hot dogs is less than 5, this cell needs to be combined with other cells. The only sensible combination would be Year 7 and Year 8.

Tip: Remember that the requirement is for the expected values to be more than 5 – it does not matter about the observed values.

b Combining year 7 and year 8 the observed values are:

	Year 7 & 8	Year 9	Year 10	Year 11
Pizza	16	9	12	8
Hot dog	10	9	10	9
Lasagne	12	8	14	15
Burger	23	10	9	7

Using the GDC, this provides a χ^2 value of 9.676 and a p-value of 0.377. Since this p value is more than 0.05 there is no significant evidence that favourite food depends on age.

c Just because there is no evidence of dependence, it does not mean that there is significant evidence of independence. The p-value means that we would only expect to see this Type of data approximately 38% of the time when the data is actually independent.

3 a Linear model: $r^2 = 0.870$

Exponential model: $r^2 = 0.983$

On this basis, the exponential model (Qingqing's) is the better fit.

b The model $y = 0.412e^{0.360x}$ predicts $y(8) = 7.36$

4 a This is a z-test with

$H_0: \mu = 10.3$

$H_1: \mu < 10.3$

From the GDC this has a z-score of -1.33 and a p value of 0.0918. Since the p-value is less than 10% this provides significant evidence that the population mean is less than 10.3.

b There are two assumptions made here. First of all we are assuming that the original measured standard deviation is the population standard deviation. If the population is large, this is a reasonable assumption. Secondly, we are assuming that the standard deviation has not changed after the typing course. This seems less reasonable as the course might have had a bigger effect on those students who were originally slower.

c We could repeat the test with a different sample and see if this also provided significant evidence of a decreased mean.

5 a From the GDC, $\bar{x} = 73.2\text{g}$ and $s_{n-1} = 11.6\text{g}$ so $s_{n-1}^2 = 134\text{g}^2$.

b From the GDC, the confidence interval (using z-interval) is (66.5, 79.9).

6 From the GDC, the confidence interval (using z-interval) is (85.8, 90.6).

7 a $H_0: \mu_1 = \mu_2, H_1: \mu_1 \neq \mu_2$

b Using a 2-sample z-test on the GDC $z = 1.29$ and the p-value is 0.197.

c Since the p-value is greater than 0.1, there is not significant evidence that the means are different.

8 **a** $H_0: \rho = 0, H_1: \rho < 0$

b Both distributions need to be normal.

c The appropriate test is the test for correlation. From the GDC, the p-value is 0.0312.

d Since the p-value is less than 0.05, there is significant evidence of negative correlation between temperature and cloud cover.

e There might be different patterns in different seasons.

9 **a** X is the random variable 'Number of bookings in 3 hours'. Then $X \sim \text{Po}(51)$.

$P(X \geq 60) = 1 - P(X \leq 59)$

But $P(X \leq 59)$ can be found using the cumulative Poisson distribution function on the calculator. So

$P(X \geq 60) = 1 - 0.881 = 0.119$

b If λ is the hourly rate, then

$H_0: \lambda = 17, H_1: \lambda > 17$

Tip: You could also have defined λ as the rate per three hours and then you would have $H_0: \lambda = 51, H_1: \lambda > 51$

c The p-value for this test is the probability found in part **a**. Since the p-value is greater than 0.05, there is not significant evidence that the number of bookings has increased.

10 The table in the first edition is incorrect. It should read:

		Grade		
		5	**6**	**7**
Revision time (hours)	**10–12**	2	16	8
	13–15	12	20	14
	16–20	14	18	18
	21–25	3	2	14

a Students might not answer honestly.

b The expected values from the GDC are:

		Grade		
		5	**6**	**7**
Revision time (hours)	**10–12**	5.72	10.33	9.96
	13–15	10.11	18.27	17.62
	16–20	10.99	19.86	19.15
	21–25	4.18	7.55	7.28

Since the expected number of people in Grade 5 who study 21–25 hours is less than five, this cell needs to be combined. Combining columns would reduce degrees of freedom more than combining rows, so this would be a sensible thing to do.

c Combining these two rows would not create a meaningful category. It is better to combined adjacent rows.

d Combining 16–20 and 21–25 produces the following observed frequencies.

		Grade		
		5	**6**	**7**
Revision time (hours)	**10–12**	2	16	8
	13–15	12	20	14
	16–25	17	20	32

From the GDC, the chi-squared statistic is 10.6 and the p-value is 0.0318. Since this is less than 10%, there is significant evidence that revision time and grade are dependent.

11 Let X be the number of heads in 30 flips of the coin. $X \sim \mathrm{B}(30, p)$

$\mathrm{H}_0: p = 0.5$

$\mathrm{H}_1: p > 0.5$

The p-value is $\mathrm{P}(\mathrm{X} \geq 19) = 0.100 > 0.05$

Do not reject H_0; there is insufficient evidence that the probability of heads is greater than 0.5.

12 $\mathrm{H}_0: \rho = 0$

$\mathrm{H}_1: \rho \neq 0$

Doing a test for correlation on the GDC we find that there is insufficient evidence of linear correlation ($p = 0.0578<0.05$)

13 a From the z-interval on the calculator the interval is 9.761 to 9.825.

b If a large number of intervals are formed in this way, 99% of them will contain μ.

14 a $\mathrm{H}_0: \rho = 0,\ \mathrm{H}_1: \rho > 0$

b From the GDC, $r = 0.853$.

c From the GDC, the p-value is 0.00173. Since this is less than 5% (the default significance level) there is sufficient evidence of positive correlation.

d From the GDC, the y-on-x regression line is $y = 1.78x + 40.5$.

e When $x = 19$, then $y = 1.78 \times 18 + 40.5 = 74.315$.

15 a $\mathrm{H}_0: \rho = 0$

$\mathrm{H}_1: \rho > 0$

From the GDC: $p = 0.0574$

b The test was for a linear relationship; that there is not a significant linear relationship does not mean there is no relationship at all; there may be an underlying non-linear relationship.

c $P = -0.202n^2 + 1.72n + 1.79$

d $R^2 = 0.979$, which suggests that the quadratic model is a good fit.

Tip: As a rule of thumb, any value of R^2 above 0.7 is considered a good fit. In exams, the cut-off you should use for a 'good fit' should be given.

16 a For Model A:

Time	Model value	Residual	Residual squared
01:00	9.74	9.74 −7.9 = 1.84	3.40
05:00	9.58	9.58 − 8.9 = 0.682	0.466

So $3.40 + 0.466 \approx 3.87$ must be added to the previous SS_{res} for model A making a total of 6.02

For Model B:

Time	Model value	Residual	Residual squared
01:00	7.46	7.46 − 7.9 = −0.444	0.197
05:00	8.36	8.36 − 8.9 = −0.537	0.288

So $0.197 + 0.288 \approx 0.485$ must be added to the previous SS_{res} for model B making a total of 5.17.

b Since Model B has a lower sum of square residuals it is a better fit, so he should choose Model B.

17 a The estimate of standard deviation is $\sqrt{144\,000\,000} = 12\,000$ (£).

Since we only have an unbiased estimate of the variance we need a t interval.

From the calculator using a t-interval the answer is (21 100, 27 900)

b Since the confidence interval includes £25 000 there is no significant evidence at the 5% significance level that the mean is different from £25 000

c The population of wages is normally distributed.

18 a From the GDC, 90% confidence t-interval is $(31.3, 53.5)$

b $\bar{x} = 28.4, s_n = 4.2, n = 8$

Tip: In real life, there is often ambiguity as to whether a reported standard deviation is the standard deviation of the sample or the best estimate of the standard deviation of the population. In this case, the more natural interpretation is that the quoted number is the standard deviation of the sample. In exams, the wording should be very clear.

$$s_{n-1} = s_n\left(\sqrt{\frac{n}{n-1}}\right) = 4.49$$

From the GDC, the 90% confidence t-interval is $25.4 < \mu < 31.4$.

c The confidence intervals scarcely overlap, which suggests that the estimates may not be reliable, or that Martin and Oli interviewed different cohorts or in different ways that materially biased the data collected.

d e.g. Increase sample size, randomise interviewees, standardise the interview question method and wording.

19 a H_0: μ=2.7; H_1: $\mu \neq 2.7$.

b We want $P(-z_{crit} < z < z_{crit}) = 0.9$.

Either using the symmetry of the normal distribution to get $P(Z < z_{crit}) = 0.95$ or if the GDC has ability to do an inverse normal calculation in a symmetric range, you find from the GDC that $z_{\text{crit}} = 1.64$.

Then

$$z_{\text{crit}} = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{x} - 2.7}{0.7/\sqrt{45}} = 1.64$$

Rearranging, $\bar{x} = 1.64 \times \frac{0.7}{\sqrt{45}} + 2.7 = 2.87$.

The other critical value is symmetrically on the other side of the hypothesised mean, 2.7:

$$\bar{x} = -1.64 \times \frac{0.7}{\sqrt{45}} + 2.7 = 2.53$$

The critical region is any point further than these calculated values from the hypothesised mean, so $\bar{x} < 2.53$ or $\bar{x} > 2.87$

c The observed value is in the rejection region so there is significant evidence that the height of the trees in the new orchard differs from the farmer's previous orchard.

d Yes, since the sample size is large enough that the Central Limit Theorem makes the mean approximately normally distributed anyway.

20 a From the calculator the mean is 7.09. This is equal to λ, the mean number of visitors per hour.

b H_0:Poisson is a suitable model

H_1:Poisson is not a suitable model

c If $X \sim \text{Po}(7.09)$, from the frequencies are found by multiplying the probabilities from the GDC by the total frequency, 90:

Number of visitors	≤ 3	4	5	6	7	8	9	10	≥ 11
Frequency	6.95	7.90	11.2	13.2	13.4	11.9	9.35	6.63	9.45

d There are 9 groups and two constraints estimated from the data (the total and the mean) so there are $9 - 7 = 2$ degrees of freedom.

e Using a chi squared test on the GDC, $\chi^2 = 27.3, p = 2.89 \times 10^{-4}$. Since the p-value is less than 0.01 there is significant evidence that the Poisson distribution is not a suitable model.

21 a $\mathrm{H_0}: p = 0.72$, $\mathrm{H_1}: p > 0.72$

b Let X be the number of pupils in a sample of 30 who eat school lunch; then, assuming $\mathrm{H_0}$ is true, $X \sim B(30, 0.72)$

$\mathrm{P}(X \geq 25) = 1 - \mathrm{P}(X \leq 24) = 0.116 > 0.05$

$\mathrm{P}(X \geq 26) = 1 - \mathrm{P}(X \leq 25) = 0.0495 < 0.05$

So the critical region is $X \geq 26$.

c The probability of Type I error is the probability of being in the critical region when $\mathrm{H_0}$ is true, so it equals 0.0495 (as found in part **b**).

22 a 24, 32, 26, 23

b Using the GDC, $\bar{x} = 26.25$, $s_{n-1} = 4.03$. The t-interval is 21.5g to 31.0g

c The differences in weights are normally distributed, as the *t*-interval requires underlying population to be normal.

23 a Let X be the number of accidents in a month and X_4 the number of accidents in a

four- month period.

$X \sim \mathrm{Po}(\lambda), X_4 \sim \mathrm{Po}(4\lambda)$

$\mathrm{H_0}: \lambda = 1.4$

$\mathrm{H_1}: \lambda < 1.4$

For the critical region $X_4 \leq c$, require the greatest integer c such that $\mathrm{P}(X_4 \leq c) < 0.1$

Doing a search on the GDC, using $X_4 \sim \mathrm{Po}(5.6)$

$\mathrm{P}(X_4 \leq 3) = 0.191$

$\mathrm{P}(X_4 \leq 2) = 0.0824$

The critical region is $X_4 \leq 2$

b $\mathrm{P}(\text{Type I error}) = \mathrm{P}(X_4 \leq 2) = 0.0824$

c $\mathrm{P}(\text{Type II error}) = \mathrm{P}(X_4 > 2 | \lambda = 1.2) = 0.857$, using $X_4 \sim \mathrm{Po}(4.8)$

24 a $\mathrm{H_0}: T$ follows a normal distribution

$\mathrm{H_1}$: T does not follow a normal distribution

b

	$]-\infty, 3[$	$[3, 7[$	$[7, 10[$	$[10, 15[$	$[15, 18[$	$[18, 20[$	$[20, 25[$	$[25, 30[$	$[30, \infty[$
T	–	5	8.5	12.5	16.5	19	22.5	27.5	–
f	0	4	12	21	26	18	7	3	0

Table shows the midpoint values as T.

Estimate of mean $\bar{T} = \frac{1}{n}\sum_i T_i f_i = 15.3$

The GDC gives $s_{n-1} = 4.99$

c

	$]-\infty, 3[$	$[3, 7[$	$[7, 10[$	$[10, 15[$	$[15, 18[$	$[18, 20[$	$[20, 25[$	$[25, 30[$	$[30, \infty[$
$\mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$	0.61	3.69	8.65	30.11	20.96	11.08	13.51	2.24	0.15

d A chi-squared test requires all expected values in cells to be at least 5. Since this is not the case in the extreme cells, they need to be merged to achieve this condition.

The amended table is as follows:

	$]-\infty, 10[$	$[10, 15[$	$[15, 18[$	$[18, 20[$	$[20, \infty[$
$\mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$	12.95	30.11	20.96	11.08	15.90

There remain only 5 cells.

Two parameters have been estimated for the model being fitted, so degrees of freedom $\nu = 5 - 3 = 2$

e

	$]-\infty, 10[$	$[10, 15[$	$[15, 18[$	$[18, 20[$	$[20, \infty[$	χ^2
$\boldsymbol{f}$	16	21	26	18	10	
$\mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$	12.95	30.11	20.96	11.08	15.90	
						11.2

$\mathrm{P}(\chi_2^2 > 11.2) = 0.00370 < 0.05$

Reject H_0; the normal distribution is not a good model for this data set.

25 a Students may not answer honestly if the teacher is asking them directly.

b $\mathrm{H}_0: p = 0.6, \mathrm{H}_1: p < 0.6$

c If the null hypothesis is true then X will be distributed B(50,0.6)

Then if x students say yes the p-value would be $P(X \leq x)$.

Investigating this on the GDC:

$\boldsymbol{x}$	$\boldsymbol{P(X \leq x)}$
21	0.007617
22	0.016035
23	0.031406
24	0.057344
25	0.097807

The largest value of x with a p-value below 5% is 23. Therefore the critical region is $X \leq 23$.

d The null hypothesis will be accepted if $X \geq 24$. If $p = 0.43$ this would be a Type II error, and the probability of it occurring, from the GDC is:

$$\begin{aligned}\mathrm{P}\big(X \geq 24 \big| X \sim B(50,0.43)\big) &= 1 - \mathrm{P}\big(X \leq 23 \big| X \sim \mathrm{B}(50,0.43)\big) \\ &= 0.283\end{aligned}$$

e

$$\mathrm{P}\big(X \geq 24 \big| X \sim B(50,0.63)\big) = 1 - \mathrm{P}\big(X \leq 23 \big| X \sim B(50,0.63)\big)$$
$$= 0.989$$

26 a $\mathrm{H}_0 : p = 0.1, \mathrm{H}_1 : p > 0.1$

Let X be the number of trains in the sample which are late.

If the null hypothesis is true then $X \sim \mathrm{B}(16,0.1)$

The p-value is $\mathrm{P}(X \geq 4) = 0.0684$ from the GDC.

This is greater than 0.05 therefore we do not reject the null hypothesis. There is no significant evidence that there are more than 10% of trains which are late.

b To find the probability of a Type I error we need to find the critical region. Checking some p-values using the GDC:

x	$P(X \geq x)$
4	0.068406
5	0.017004
6	0.003297

The smallest value of x which has a p-value less than 0.05 is 5; therefore the critical region is $X \geq 5$.

A Type I error occurs when x is 5 or more when it follows a B(16,0.1) distribution. From the table above, the probability of this occurring is 0.0170.

c If the true value of p is 0.18 then accepting the null hypothesis is a Type II error. The probability of this occurring is

$\mathrm{P}(X \leq 4 | X \sim \mathrm{B}(16,0.18) = 0.854$

d This is the probability of not making a Type II error which is $1 - 0.854 = 0.146$.

27 a Let X be the number of brown eggs in a box. From the GDC, $\bar{x} = 2.4$

If $X \sim \mathrm{B}(6,p)$ then, by equating the sample mean and the formula for expectation, we can estimate estimate $6p = 2.4$; therefore $p = 0.4$.

b H_0: Number of brown eggs follows a distribution $\mathrm{B}(6, 0.4)$

H_1: Number of brown eggs does not follow a distribution $\mathrm{B}(6, 0.4)$

x	0	1	2	3	4	5	6	χ^2_3
f	7	32	35	50	22	4	0	
$\mathbf{B(6, 0.4)}$	7.0	28.0	46.7	41.5	20.7	5.5	0.6	

Since the final column has an expected value below 5, merge the final two columns.

x	0	1	2	3	4	5 or 6	χ^2_4
f	7	32	35	50	22	4	
$\mathbf{B(6, 0.4)}$	7.0	28.0	46.7	41.5	20.7	6.1	
							6.06

One parameter has been estimated from the data, so degrees of freedom $\nu = 6 - 2 = 4$

$\mathrm{P}(\chi^2_4 > 6.06) = 0.194 > 0.05$

Cannot reject H_0; the farmer's claim is consistent with the data.

28 a $\mathrm{H}_0: \mu = 2.5, \mathrm{H}_1: \mu \neq 2.5$

b The test statistic $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ follows a normal distribution $\mathrm{N}(0,1)$. For a two tailed test at 5% significance we want

$\mathrm{P}(-z^* < z < z^*) = 0.95$

Using the symmetry of the normal distribution, this means that

$\mathrm{P}(z < z^*) = 0.975$

Using the inverse normal distribution this means that

$z^* = 1.96$

Then at the boundaries of the critical region either

$$\frac{\bar{x} - 2.5}{0.1/\sqrt{16}} = 1.96$$

or

$$\frac{\bar{x} - 2.5}{0.1/\sqrt{16}} = -1.96$$

So the boundaries are $\bar{x} = 2.55$ and 2.41.

The critical region is therefore $\bar{x} < 2.45$ or $\bar{x} > 2.55$

c If the true mean is 2.6 then accepting the null hypothesis would be a Type II error. The probability of this occurring would be

$$\mathrm{P}\left(2.45 < x < 2.55 \middle| x \sim \mathrm{N}\left(2.6, \frac{0.1}{\sqrt{16}}\right)\right) = 0.0228$$

Tip: This answer was found using the 3 s.f. rounded critical region described in part **b**. Best practice would be to use unrounded numbers stored on your calculate of 2.451 00… to 2.548 99…

If you do this, the answer becomes 0.0207. Any answer in this range would be acceptable in an exam.

29 a i $X \sim \mathrm{N}(\mu, 0.25)$

$\mathrm{H}_0: \mu = 3$

$\mathrm{H}_1: \mu < 3$ (one tail)

1 tailed z test

Require A such that $\mathrm{P}(\bar{X} < A) \leq 0.05$

$A = 3 - z_{0.95}\frac{\sigma}{\sqrt{n}} = 3 - 1.645\frac{\sqrt{0.25}}{6} = 2.863$

The condition is $\bar{x} < 2.863$

ii Type I error is rejecting H_0 when it is true.

iii Type II error is failing to reject H_0 when it is false.

iv $\mathrm{P}(\text{Type I error}) = \text{significance} = 0.05$

v $\mathrm{P}\left(\bar{X} > 2.863 \middle| X \sim \mathrm{N}(2.75, 0.25)\right) = 0.0877$

b i One sample t-test

ii $\mathrm{H}_0: \mu = 3$

$\mathrm{H}_1: \mu < 3$ (one tail)

$\bar{y} = 2.860, s_{n-1}^2 = 0.25, n = 36$

From the GDC, $p = 0.0509 > 0.05$

Do not reject H_0; the data are consistent with waves of mean height 3 m.

iii 90% confidence t-interval is

$\bar{x} - t_{35}\frac{s_{n-1}}{\sqrt{n}} < \mu < \bar{x} + t_{35}\frac{s_{n-1}}{\sqrt{n}}$

$2.719 < \mu < 3.001$

30 a Using the cubic regression function on the calculator:

$p = -0.000964y^3 + 0.111y^2 - 3.43y + 151$

$R^2 = 0.680$

b $p = -0.000140w^3 + 0.0628w^2 - 6.72w + 332$

$R^2 = 0.767$

c Using R^2, Nathan's model accounts for 76.7% of the variability in p whereas Marc's only accounts for 68.0%.

d A more complex model where p depends on both weight and age might be superior.

31 a $T = 4.21Q^2 - 44.9Q + 656$

b $R^2 = 0.973$

c i Model predicts $T(10) = 627 \approx \$630$.

ii Model predicts $T(40) = 5596 \approx \$5699$.

d **ci** is interpolation on a curve with good fit and so is reliable.

cii is extrapolation beyond the data and is therefore unreliable

e $T = 0.257Q^2 + 31.7Q + 192$

$R^2 = 0.935$

f The sum of residuals is not appropriate because the two data sets have different numbers of data points.

g $R^2_{\text{Irina}} = 0.973, R^2_{\text{Julian}} = 0.935$ suggesting that Irina's model is a better fit to her data than Julian's is to his.

h The R^2 value for the fitted cubic is 0.993, which is greater than the quadratic 0.973 but the claim is spurious, because the new model has an extra parameter so the comparison is unsuitable.

Comment: A (four-parameter) cubic must always be at least as good as a (three-parameter) quadratic because the quadratic is a cubic with first coefficient zero and so the *best fit* cubic cannot be worse than this. Extending the logic, using $n - 1$ parameters, a perfect fit can be found for any data set with non-repeating x values. However, R^2 is not the only measure of how good a model is – high R^2 with few parameters is the ideal. Each added parameter must materially improve the model.

32 a Assume that $X \sim \mathrm{N}(\mu, 0.3^2)$

Since the hypothetical mean must lie at the centre of the acceptance region, $\mu = 6.4$.

$\mathrm{H}_0: \mu = 6.4$

$\mathrm{H}_1: \mu \neq 6.4$

b $\mathrm{P}(X > 6.542) = 0.9495$ so each tail carries 5.05%

The significance level of the test is 10%.

33 a Test B is better, since it is a paired sample test and so eliminates variation due to an individual athlete's performance, which allows clearer investigation of the difference due to diet.

b

Athlete	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
d	−0.6	−1.1	−0.1	0.2	−0.3	0.3	−0.4	0.2	−0.4	−1.2

$\bar{d} = -0.34$

$s_n = 0.494$

c $\mathrm{H}_0: \mu_D = 0$

$\mathrm{H}_1: \mu_D < 0$

One sample (paired) t-test: $p = 0.312$, greater than any reasonable significance level.

Fail to reject H_0; there is insufficient evidence of improvement.

34 Let X be the number of cars passing through each day. $X \sim \text{Po}(23)$.

Let Y be the number of lorries passing through each day. $Y \sim \text{Po}(8)$.

Let V be the total number of cars and lorries passing through each day. $V \sim \text{Po}(31)$.

Then let V_7 be the number of cars and lorries in a week: $V_7 \sim \text{Po}(217)$.

a Require least c such that $\text{P}(V_7 \geq c) < 0.05$

$\text{P}(\text{V}_7 \geq 242) = 0.05005$

$\text{P}(\text{V}_7 \geq 243) = 0.04365$

The critical region is $V_7 \geq 243$

b The test on cars:

Require least k_X such that $\text{P}(X_7 > k_X) < 0.025$

$\text{P}(X_7 \geq 186) = 0.02887$

$\text{P}(X_7 \geq 187) = 0.02427$

The critical region is $X_7 \geq 187$

The test on cars:

Require least k_X such that $\text{P}(X_Y > k_Y) < 0.025$

$\text{P}(Y_7 \geq 71) = 0.02982$

$\text{P}(Y_7 \geq 72) = 0.02239$

The critical region is $Y_7 \geq 72$

The probability of a Type I error for Aline is 0.04365.

The probability of a Type I error for Reto is
$1 - (1 - 0.02239)(1 - 0.02427) = 0.04611$.

So Reto has the larger probability of a Type I error.

35 a $\text{H}_0: \mu = 1.2, \text{H}_1: \mu < 1.2$

b i If X is the number of breakdowns in 30 days, then $X \sim \text{Po}(30 \times 1.2)$.

The significance level is the probability of rejecting the null hypothesis when it is true (a Type I error). From the GDC, this is

$$\text{P}\big(X \leq 25 \big| X \sim \text{Po}(36)\big) = 0.0345$$

ii The number of breakdowns in 30 days now follows a $\text{Po}(22.5)$ distribution.

A Type II error would be accepting the null hypothesis now that we know it is false. This will happen if $x \geq 26$. So we want

$$\text{P}\big(X \geq 26 \big| X \sim \text{Po}(22.5)\big) = 1 - P\big(X \leq 25 \big| X \sim \text{Po}(22.5)\big)$$

$$= 0.257$$

Worked solutions

36 a Let S be the sum of the 10 observations of X. Then if the null hypothesis is true $S \sim \text{Po}(1 \times 10)$. A Type I error would be rejecting H_0 when it is true. The probability of this happening is:

$$\text{P}\big(S \geq 15 \big| S \sim \text{Po}(10)\big) = 1 - \text{P}\big(S \leq 14 \big| S \sim \text{Po}(10)\big)$$
$$= 0.0835$$

b A Type II error is when $\mu = 2$ but we accept that the null hypothesis is true. In this case $S \sim \text{Po}(2 \times 20)$ and the error occurs when

$$\text{P}\big(S \leq 14 \big| S \sim \text{Po}(20)\big) = 0.105$$

37 a $A \sim \text{N}(5.2, 1.2^2)$

If $Z \sim \text{N}(0,1)$, then $\text{P}(Z < z^*) = 0.05$ is satisfied by $z^* = -1.64$ (using the inverse normal function on the GDC). Since the usual test statistic follows this standard normal distribution this means that at the boundary of the critical region

$$\frac{\bar{X} - 5.2}{1.2/\sqrt{16}} = -1.64$$

Rearranging this gives$\bar{x} = 5.69$ so the critical region is $\bar{X} < 4.71$

b The critical region is still as found in (a). Now, accepting the null hypothesis would be a Type II error. This occurs with probability

$$P\left(X > 4.71 \middle| X \sim N\left(4.6, \frac{1.2^2}{16}\right)\right) = 0.361$$

Tip: The above answer used the exact value from part **b**. If you used the 3sf answer you would have got an answer of 0.357.

$\text{B} \sim \text{N}(4.6, 1.2^2)$

c There are two ways of the mean falling into the critical region – either when species A or species B. You can think about this using a tree diagram, to get:

$$\begin{aligned}\text{P}(\bar{X} < 4.71) &= \text{P}(\bar{X} < 4.71|A)P(A) + \text{P}(\bar{X} < 4.71)\text{P}(B)\\ &= 0.05 \times 0.9 + 0.063876 \times 0.1\\ &= 0.109\end{aligned}$$

d We want $\text{P}(A|\bar{X} < 4.71)$. Using the formula for conditional probability this is

$$\frac{\text{P}(A \text{ AND } \bar{X} < 4.71)}{\text{P}(\bar{X} < 4.71)} = \frac{0.05 \times 0.9}{0.109}$$

Where the denominator is found from part **c**.

This evaluates to 0.413.

10 Differentiation

Worked solutions

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 10A

17 $y = \sin x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{\pi}{3}} = \frac{1}{2}$$

18 $h = 10t^{0.5} + 20t^{-0.5}$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 5t^{-0.5} - 10t^{-1.5}$$

$$\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{t=1} = 5 - 10 = -5$$

The sign of height change is negative, so the falcon is descending.

Its rate is 5 m s^{-1}.

19 $y = 2 + 5x - \mathrm{e}^x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - \mathrm{e}^x = 4$$

$$\mathrm{e}^x = 1$$

So $x = 0$

$$y|_{x=0} = 1$$

The point with gradient 4 is $(0,1)$

20 $y = \mathrm{e}^x + 3x + 4$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x + 3$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\ln 2} = 2 + 3 = 5$$

21 $y = 3\cos x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3\sin x$$

Gradient of $2y + 3x + 3 = 0$ is $-\frac{3}{2}$

$$-3\sin x = -\frac{3}{2}$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}$$
$$y|_{x=\frac{\pi}{6}} = \frac{3\sqrt{3}}{2}$$

The tangent has equation:

$$y - \frac{3\sqrt{3}}{2} = -\frac{3}{2}\left(x - \frac{\pi}{6}\right)$$
$$y = -\frac{3}{2}x + \frac{\pi}{4} + \frac{3\sqrt{3}}{2}$$

22 $y = \mathrm{e}^x - 2x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - 2$$

Gradient of $y = -x + 4$ is -1 so the perpendicular has gradient 1.

$$\mathrm{e}^x - 2 = 1$$
$$x = \ln 3$$

When $x = \ln 3$, $y = 3 - 2\ln 3$

The perpendicular has equation:

$$y - 3 + 2\ln 3 = x - \ln 3$$
$$y = x + 3 - 3\ln 3$$

23 $y = x^{0.5}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.5x^{-0.5}$$

Gradient of $y = \frac{x}{6} + \frac{3}{2}$ is $\frac{1}{6}$

$$0.5x^{-0.5} = \frac{1}{6}$$
$$\sqrt{x} = 3$$
$$x = 9$$
$$y|_{x=9} = 3$$

The point of tangency is $(9, 3)$

24 a $y = 2x - e^x$

$$\frac{dy}{dx} = 2 - e^x$$
$$\frac{dy}{dx} = 2 - e^x$$
$$\left.\frac{dy}{dx}\right|_{x=\ln 2.5} = 2 - 2.5 = -0.5$$

Tangent gradient is -0.5 so the normal gradient is 2.

$$y|_{x=\ln 2.5} = 2\ln 2.5 - 2.5$$

Normal equation is:

$$y + 2.5 - 2\ln 2.5 = 2(x - \ln 2.5)$$
$$y = 2x - 2.5$$

b Intersection of the original curve and the normal:

$$2x - e^x = 2x - 2.5$$
$$e^x = 2.5$$

This has only one solution, at $x = \ln 2.5$, so this is the only point of intersection.

25 $y = \tan x$

$$\frac{dy}{dx} = \sec^2 x$$
$$\left.\frac{dy}{dx}\right|_{x=k} = \sec^2 k$$
$$y|_{x=k} = \tan k$$

Tangent has equation:

$$y - \tan k = \sec^2 k\,(x - k)$$
$$y = \frac{1}{\cos^2 k}(x - k + \sin k \cos k)$$

26 a $N(0) = 1$ so the initial population is 1 million

b

$$N = 16 = \left(1 + \sqrt{t}\right)^2$$
$$\sqrt{t} = 3$$
$$t = 9$$

It takes 9 hours for the population to reach 16 million.

c $N = 1 + 2t^{0.5} + t$

$$\frac{dN}{dt} = t^{-0.5} + 1$$
$$\left.\frac{dN}{dt}\right|_{t=4} = 1.5$$

Rate of increase at 4 hours is 1.5 million per hour.

d The model projects no upper limit to population size, though in reality there would be constraints including space, accumulation of waste and limitations of food resources.

27 $y = \ln x - 2x$

$$\frac{dy}{dx} = \frac{1}{x} - 2$$

The function is decreasing where $\frac{dy}{dx} < 0$

$$\frac{1}{x} - 2 < 0$$
$$x > \frac{1}{2}$$

28 $y_1 = x^{0.5}$

$$\frac{dy_1}{dx} = 0.5x^{-0.5}$$

The tangent at $x = k$ has gradient $\frac{1}{2\sqrt{k}}$

$$y_2 = x^{-0.5}$$
$$\frac{dy_2}{dx} = -0.5x^{-1.5}$$

The tangent at $x = k$ has gradient $-\frac{1}{2k\sqrt{k}}$

If these two tangents are perpendicular then the product of their gradients is -1.

$$\frac{1}{2\sqrt{k}} \times \left(-\frac{1}{2k\sqrt{k}}\right) = -1$$
$$-\frac{1}{4k^2} = -1$$
$$k = \pm\frac{1}{2}$$

Since k must be in the domain of $y = \sqrt{x}$, only the positive solution is valid.

$$k = \frac{1}{2}$$

29 a $P = \left(\frac{\pi}{2}, 0\right), Q = \left(-\frac{\pi}{2}, 0\right)$

$$y = \cos x$$
$$\frac{dy}{dx} = -\sin x$$
$$\left.\frac{dy}{dx}\right|_{x=\pm\frac{\pi}{2}} = \pm 1$$

Since the curve is even (symmetrical about $x = 0$), the tangents will be reflections of each other and will intersect at $x = 0$.

Tangent at P has gradient -1 and passes through P.

Tangent has equation:

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$
$$y = -x + \frac{\pi}{2}$$

29 a (continued)

The y-intercept is the intersection of the tangent lines $\left(0, \frac{\pi}{2}\right)$

The width of the triangle is π and its height is $\frac{\pi}{2}$ so the area is $\frac{\pi^2}{4}$

b The sine curve will behave in the same way as the cosine curve, so the triangle produced will have the same area, $\frac{\pi^2}{4}$.

30 $y = x^{0.5}$

$$\frac{dy}{dx} = 0.5x^{-0.5}$$

The tangent at point (a^2, a) has gradient $\frac{1}{2a}$ and equation:

$$y - a = \frac{1}{2a}(x - a^2)$$
$$y = \frac{1}{2a}x + \frac{a}{2}$$

The y-intercept at (0, 1) is also $\left(0, \frac{a}{2}\right)$.

$$\frac{a}{2} = 1$$
$$a = 2$$

The point is $(4, 2)$.

31 $y = \sin x - \frac{1}{2}$

$$\frac{dy}{dx} = \cos x$$

The gradient of $2\sqrt{3y} = 3x - \frac{13\pi}{2}$ is $\frac{\sqrt{3}}{2}$

$\cos x = \frac{\sqrt{3}}{2}$ has solutions $x = \pm\frac{\pi}{6} + 2n\pi$

At these points, $\sin x - \frac{1}{2} = 0 \text{ or } -1$

The line passes $x = 0$ at $\left(\frac{13\pi}{6}, 0\right)$

The line passes $x = -1$ at $\left(\frac{13\pi}{6} - \frac{2}{\sqrt{3}}, -1\right)$ which is not on the curve.

P has coordinates $\left(\frac{13\pi}{6}, 0\right)$

32 Gradient of $y + 3x = 9$ is -3

$$y = x^2 - 2\ln x$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{2}{x} = \frac{2}{x}(x^2 - 1)$$

Require that the gradient equals -3.

$$\frac{2}{x}(x^2 - 1) = -3$$
$$2x^2 + 3x - 2 = 0$$
$$(2x - 1)(x + 2) = 0$$
$$x = \frac{1}{2} \text{ or } -2$$

Since -2 is not in the domain of the original function, there is only one point whose tangent is parallel to the given line, at $x = 0.5$.

$$\left(\frac{1}{2}, \frac{1}{4} - \ln\frac{1}{4}\right)$$

Exercise 10B

15 $\mathrm{f}(x) = \mathrm{e}^{-x}$

$\mathrm{f}'(x) = -\mathrm{e}^{-x}$

16 $y = \mathrm{e}^{2x} + 5x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x} + 5$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\ln 3} = 2 \times 9 + 5 = 23$$

17 $y = (x^2 + 9)^{0.5}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0.5(2x)(x^2 + 9)^{-0.5}$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=4} = \frac{(0.5)(8)}{\sqrt{25}} = 0.8$$
$$y|_{x=4} = \sqrt{25} = 5$$

Tangent equation is:

$$y - 5 = 0.8(x - 4)$$
$$y = 0.8x + 1.8$$

18 Gradient of $y = 2x$ is 2

$$y = \ln(2x - 5)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x - 5} = 2$$
$$y|_{x=3} = 0$$

The point of tangency is $(3,0)$

19 $y = \cos(x^{-1})$

$$\frac{dy}{dx} = -x^{-2}(-\sin x^{-1}) = \frac{\sin(x^{-1})}{x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{2}{\pi}} = \frac{\pi^2}{4}\sin\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$y|_{x=\frac{2}{\pi}} = \cos\left(\frac{\pi}{2}\right) = 0$$

Tangent has equation:

$$y = \frac{\pi^2}{4}\left(x - \frac{2}{\pi}\right)$$

$$y = \frac{\pi^2}{4}x - \frac{\pi}{2}$$

20 $y = \ln(x-2)$

$$\frac{dy}{dx} = \frac{1}{x-2}$$

Gradient of $y = -3x + 2$ is -3 so perpendicular gradient is $\frac{1}{3}$

$$\frac{1}{x-2} = \frac{1}{3}$$

$$x = 5$$

When $x = 5, y = \ln 3$

Tangent is:

$$y - \ln 3 = \frac{1}{3}(x-5)$$

$$y = \frac{1}{3}x + \ln 3 - \frac{5}{3}$$

21 **a** $y = \sin^2 x$

$$\frac{dy}{dx} = 2\cos x \sin x = \sin 2x$$

b

$$\sin 2x = 0$$

$$2x = n\pi$$

$$x = \frac{n\pi}{2}$$

$$x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

22 **a**

$$\begin{aligned}(\cos x + \sin x)^2 &= \cos^2 x + 2\sin x\cos x + \sin^2 x \\ &= (\cos^2 x + \sin^2 x) + 2\sin x \cos x \\ &= 1 + \sin 2x\end{aligned}$$

22 b Differentiating both sides:

$$2(-\sin x+\cos x)(\cos x+\sin x)=2\cos 2x$$
$$\cos 2x=\cos^2 x-\sin^2 x$$

23 $\sin 3x=3\sin x-4\sin^3 x$

Differentiating both sides:

$$\begin{aligned}3\cos 3x&=3\cos x-12\cos x\sin^2 x\\&=3\cos x-12\cos x\,(1-\cos^2 x)\\&=12\cos^3 x-9\cos x\\\cos 3x&=4\cos^3 x-3\cos x\end{aligned}$$

24 $y=\mathrm{e}^{kx}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=k\mathrm{e}^{kx}$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{1}{k}}=k\mathrm{e}$$
$$y|_{x=\frac{1}{k}}=\mathrm{e}$$

Tangent L has equation:

$$y-\mathrm{e}=k\mathrm{e}\left(x-\frac{1}{k}\right)$$
$$y=k\mathrm{e}x$$

The tangent line passes through the origin for all $k>0$.

25 $y=\ln(x^2-8)$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x}{x^2-8}$$

Require tangent gradient to be 6.

$$\begin{aligned}\frac{2x}{x^2-8}&=6\\6x^2-2x-48&=0\\3x^2-x-24&=0\\(3x+8)(x-3)&=0\\x&=-\frac{3}{8}\text{ or }3\end{aligned}$$

Domain of y is $|x|>\sqrt{8}$ so reject the first root.

$x=3$

26 a $y=\mathrm{e}^{2x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}^{2x}=2y$$

b No. Any function of the form $y=A\mathrm{e}^{2x}$ will have this property for any real value A, not just $A=1$.

Exercise 10C

13

$$y = \frac{\sin x}{x} = \frac{u}{v}$$
$u = \sin x$ so $u' = \cos x$
$v = x$ so $v' = 1$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{x\cos x - \sin x}{x^2}$$
$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = -\frac{1}{\left(\frac{\pi}{2}\right)^2} = -\frac{4}{\pi^2}$$
$$y|_{x=\frac{\pi}{2}} = \frac{1}{\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}$$

Tangent has equation:

$$y - \frac{2}{\pi} = -\frac{4}{\pi^2}\left(x - \frac{\pi}{2}\right)$$
$$y = \frac{4}{\pi} - \frac{4}{\pi^2}x$$

14

$y = x\cos 2x = uv$
$u = x$ so $u' = 1$
$v = \cos 2x$ so $v' = -2\sin 2x$

$$\frac{dy}{dx} = u'v + uv' = \cos 2x - 2x\sin 2x$$
$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = -\frac{\pi}{2} \text{ so normal gradient is } \frac{2}{\pi}$$
$$y|_{x=\frac{\pi}{4}} = 0$$

Normal has equation:

$$y = \frac{2}{\pi}\left(x - \frac{\pi}{4}\right)$$

15

$f(x) = xe^{2x} = uv$
$u = x$ so $u' = 1$
$v = e^{2x}$ so $v' = 2e^{2x}$

$f'(x) = u'v + uv' = e^{2x} + 2xe^{2x}$
$f'(3) = e^6 + 6e^6 = 7e^6$

16 $y = xe^x \ln x$

$$\frac{dy}{dx} = e^x \ln x + xe^x \ln x + e^x$$
$$= e^x\left((x+1)\ln x + 1\right)$$

17 $y = \mathrm{e}^{x \sin x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (\sin x + x\cos x)\mathrm{e}^{x \sin x}$$

18 $y = \mathrm{e}^{3x} - 11x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^{3x} - 11 = 13$$

$$x = \frac{1}{3}\ln 8 = \ln 2$$

$$y|_{x=\ln 2} = 8 - 11\ln 2$$

Point of tangency is $(\ln 2, 8 - 11\ln 2)$.

19 $y = x\mathrm{e}^{-x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-x} - x\mathrm{e}^{-x} = 0$$

$$\mathrm{e}^{-x}(1 - x) = 0$$

$$x = 1$$

$$y|_{x=1} = \mathrm{e}^{-1}$$

Point is $(1, \mathrm{e}^{-1})$.

20 $y = x\ln x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ln x + 1 = 2$$

$$x = e$$

$$y|_{x=\mathrm{e}} = \mathrm{e}$$

Point is (e, e).

21

$$\begin{aligned} \mathrm{f}(x) &= x(k + x^2)^{-0.5} \\ \mathrm{f}'(x) &= (k + x^2)^{-0.5} - x^2(k + x^2)^{-1.5} \\ &= (k + x^2 - x^2)(k + x^2)^{-1.5} \\ &= k(k + x^2)^{-1.5} \end{aligned}$$

22

$$\mathrm{f}(x) = x(2 + x)^{0.5} = uv$$

$$u = x \text{ so } u' = 1$$

$$v = (2 + x)^{0.5} \text{ so } v' = 0.5(2 + x)^{-0.5}$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = u'v + uv' &= (2 + x)^{-0.5} + 0.5x(2 + x)^{-0.5} \\ &= (2 + x + 0.5x)(2 + x)^{-0.5} \\ &= 0.5(4 + 3x)(2 + x)^{-0.5} \\ &= \frac{4 + 3x}{2\sqrt{2 + x}} \end{aligned}$$

$a = 3, b = 4$

23

$$f(x) = \frac{x^2}{1+x} = \frac{u}{v}$$
$$u = x^2 \text{ so } u' = 2x$$
$$v = 1 + x \text{ so } v' = 1$$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{2x(1+x) - x^2}{(1+x)^2}$$
$$= \frac{2x + x^2}{(1+x)^2}$$
$$= \frac{x(x+2)}{(1+x)^2}$$

The denominator is always positive for $x \neq -1$.

The numerator is a positive quadratic with roots at 0 and -2, so is positive outside the roots.

The gradient is therefore positive for $x < -2$ or $x > 0$.

$a = 0, b = -2$

24 **a**

$$P = \frac{4}{1 + 3e^{-n}} = 4(1 + 3e^{-n})^{-1}$$
$$\frac{dP}{dn} = -4(-3e^{-n})(1 + 3e^{-n})^{-2}$$
$$= \frac{12e^{-n}}{(1 + 3e^{-n})^2}$$
$$= \frac{12e^{n}}{(e^{n} + 3)^2}$$

Since numerator and denominator are both always positive, the population is always growing.

b **i** $P(0) = 1$ so the initial population is 1000 rabbits.

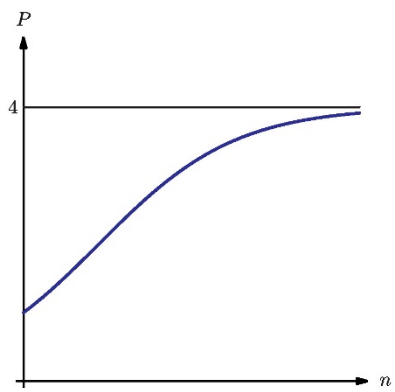
ii

$$\frac{dP}{dn}(0) = \frac{12}{4^2} = 0.75$$

The initial rate of population growth is 750 rabbits per year.

c As $n \to \infty, P \to 4$

d



25 a For $x > 0$, $x = \mathrm{e}^{\ln x}$

So $x^x = \left(\mathrm{e}^{\ln x}\right)^x = \mathrm{e}^{x\ln x}$

b $y = x^x = \mathrm{e}^{x\ln x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1 + \ln x)\mathrm{e}^{x\ln x}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = \mathrm{e}^0 = 1$$

$$y|_{x=1} = \mathrm{e}^0 = 1$$

So the tangent is:

$$y - 1 = x - 1$$
$$y = x$$

Exercise 10D

5 $A = x^2 + y^2$

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= 2(3)(1) + 2(4)(-1)\\ &= -2\end{aligned}$$

6 $B = x^3 + y^3$

$$\begin{aligned}\frac{\mathrm{d}B}{\mathrm{d}t} &= 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= 3(1)^2(1) + 3(2)^2(-2)\\ &= -21\end{aligned}$$

7 $C = xy^{-1}$

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}t} &= y^{-1}\frac{\mathrm{d}x}{\mathrm{d}t} - xy^{-2}\frac{\mathrm{d}y}{\mathrm{d}t}\\ &= \frac{3}{4} - \frac{3}{16}(-1) = \frac{15}{16}\end{aligned}$$

8 $A = x^2$

When $A = 25, x = 5$

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= 2x\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= 2(5)(2)\\ &= 20\text{cm s}^{-1}\end{aligned}$$

9 $A = \pi r^2$

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= 2\pi r\frac{\mathrm{d}r}{\mathrm{d}t}\\ &= 2\pi(3)(1.2)\\ &= 7.2\pi\ \text{mm}^2\text{ per day}\\ &= 22.6\ \text{mm}^2\text{ per day}\end{aligned}$$

10 $V=\frac{4}{3}\pi r^3$

When $V=100$, $r=\left(\frac{300}{4\pi}\right)^{\frac{1}{3}}$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$
$$\frac{dr}{dt}=\frac{1}{4\pi r^2}\frac{dV}{dt}$$
$$=1.92\text{ cm s}^{-1}$$

11 a $A=xy$

$$\frac{dA}{dt}=x\frac{dy}{dt}+y\frac{dx}{dt}$$
$$=3(-2)+4(4)$$
$$=10\text{ cm}^2\text{ s}^{-1}$$

b The diagonal length $l=\sqrt{x^2+y^2}=(x^2+y^2)^{0.5}$

$$\frac{dl}{dt}=\frac{1}{2}\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right)(x^2+y^2)^{-0.5}$$
$$=\frac{\frac{1}{2}\big(2(3)(4)+2(4)(-2)\big)}{\sqrt{25}}$$
$$=0.8\text{ cm s}^{-1}$$

12 a $V=\frac{1}{3}\pi r^2h$

At each moment of filling, the cone of water is similar to the whole cone, for which $r=5$ and $h=30$, so $r=\frac{h}{6}$.

$$V=\frac{1}{3}\pi\left(\frac{h}{6}\right)^2h=\frac{\pi h^3}{108}$$

b

$$\frac{dV}{dt}=\frac{\pi h^2}{36}\frac{dh}{dt}$$
$$\frac{dh}{dt}=\frac{36}{\pi h^2}\frac{dV}{dt}$$
$$=\frac{36}{\pi(18)^2}\times 5$$
$$=\frac{5}{9\pi}\text{ cm s}^{-1}$$

13 $A=\pi r^2$

$$\frac{dA}{dt}=2\pi r\frac{dr}{dt}$$
$$r=\frac{\frac{dA}{dt}}{2\pi\frac{dr}{dt}}=\frac{86.5}{2\pi(1.8)}=7.65\text{ cm}$$

14 If x and y are respectively the horizontal and vertical distance from the sportsman, and l is the distance from the sportsman, then $l = \sqrt{x^2 + y^2} = (x^2 + y^2)^{0.5}$

At the instant described, $\frac{\mathrm{d}x}{\mathrm{d}t} = 3, \frac{\mathrm{d}y}{\mathrm{d}t} = 0, x = 4, y = 2$

$$\begin{aligned}\frac{\mathrm{d}l}{\mathrm{d}t} &= \frac{1}{2}\left(2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t}\right)(x^2 + y^2)^{-0.5} \\ &= \frac{\frac{1}{2}\big(2(4)(3) + 2(2)(0)\big)}{\sqrt{20}} \\ &= \frac{12}{\sqrt{20}} \\ &= \frac{6}{\sqrt{5}} = 2.68 \text{ m s}^{-1}\end{aligned}$$

15 $D = mV^{-1}$

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}t} &= V^{-1}\frac{\mathrm{d}m}{\mathrm{d}t} - mV^{-2}\frac{\mathrm{d}V}{\mathrm{d}t} \\ &= V^{-1}(-2) - \frac{m}{V^2}(-1) \\ &= -2V^{-1} + \left(\frac{m}{V}\right)V^{-1} \\ &= -2V^{-1} + DV^{-1} \\ &= 3V^{-1}\end{aligned}$$

Since $V > 0$, the rate of change in density is positive; the density is increasing.

16 Let x be the distance between the base of the wall and the foot of the ladder, and y be the height above ground level of the top of the ladder.

The constant 3 m length of the ladder means that, if the ladder stays in contact with both wall and ground, $x^2 + y^2 = 9$ at all times.

$$\begin{aligned}x^2 + y^2 &= 9 \\ 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{y}{x}\frac{\mathrm{d}y}{\mathrm{d}t} = 0\end{aligned}$$

When $y = 2, \frac{\mathrm{d}y}{\mathrm{d}t} = -0.1, x = \sqrt{5}$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{2}{\sqrt{5}}(-0.1) = 0.0894 \text{ m s}^{-1}$$

Exercise 10E

13

$$\begin{aligned}\mathrm{f}(x) &= x^3 + kx^2 + 3x + 1 \\ \mathrm{f}'(x) &= 3x^2 + 2kx + 3 \\ \mathrm{f}''(x) &= 6x + 2k \\ \mathrm{f}''(1) &= 6 + 2k = 10 \\ k &= 2\end{aligned}$$

14

$$f(x) = x^3 + ax^2 + bx + 1$$
$$f'(x) = 3x^2 + 2ax + b$$
$$f''(x) = 6x + 2a$$

$$\begin{cases} f'(1) = 3 + 2a + b = 4 & (1) \\ f''(-1) = -6 + 2a = -4 & (2) \end{cases}$$

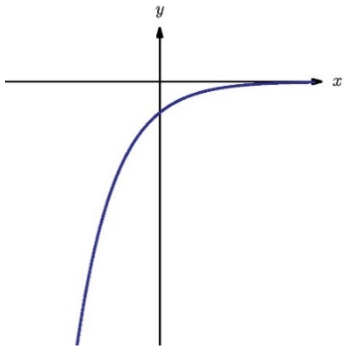
Using (2) gives $a = 1$

Substituting into (1) gives $5 + b = 4$ so $b = -1$

15 $y = xe^x$

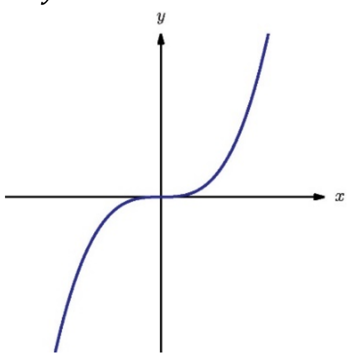
$$\frac{dy}{dx} = (x+1)e^x$$
$$\frac{d^2y}{dx^2} = (x+2)e^x$$

16 For example: $y = -e^{-x}$



17 For example: $y = x^3$

18 a $y = x^3 - kx^2 + 4x + 7$



$$\frac{dy}{dx} = 3x^2 - 2kx + 4$$
$$\frac{d^2y}{dx^2} = 6x - 2k$$

The graph is concave up when the second derivative is positive.

$6x - 2k > 0$ when $x > 1$

$k = 3$

b At $x = 1$, the second derivative is zero, so this is a point of inflection.

19

$$\begin{aligned} f(x) &= x^n \\ f'(x) &= nx^{n-1} \\ f''(x) &= n(n-1)x^{n-2} \\ f''(1) &= n(n-1) = 12 \\ n^2 - n - 12 &= 0 \\ (n-4)(n+3) &= 0 \end{aligned}$$

$n = 4 \text{ or } -3$

20

$$\begin{aligned} y &= x^2 + bx + c \\ y' &= 2x + b \\ y'(1) &= 0 = 2 + b \text{ so } b = -2 \\ y(1) &= 2 = 1 + b + c \text{ so } c = 1 - b = 3 \end{aligned}$$

21

$$\begin{aligned} y &= x^4 + bx + c \\ y' &= 4x^3 + b \\ y'(1) &= 0 = 4 + b \text{ so } b = -4 \\ y(1) &= -2 = 1 + b + c \text{ so } c = -3 - b = 1 \end{aligned}$$

22

$$\begin{aligned} y &= x - 2x^{0.5} \\ y' &= 1 - x^{-0.5} \end{aligned}$$

Stationary points where $y' = 0$ so $x = 1$

$$\begin{aligned} y(1) &= 1 - 2 = -1 \\ y'' &= 0.5x^{-1.5} \\ y''(1) &= 0.5 > 0 \end{aligned}$$

So $(1, -1)$ is a local minimum.

23

$$\begin{aligned} y &= x^2 + bx + c \\ y' &= 2x + b \\ y'' &= 2 > 0 \text{ for all } x \end{aligned}$$

Since the second derivative is always positive, the curve is concave-up for all values of x.

24

$$\begin{aligned} y &= \sin 3x + 2\cos 3x \\ y' &= 3\cos 3x - 6\sin 3x \\ y'' &= -9\sin 3x - 18\cos 3x = -9y \end{aligned}$$

25

$$\begin{aligned} y &= \ln(a + x) \\ y' &= \frac{1}{a + x} = (a + x)^{-1} \\ y'' &= -(a + x)^{-2} \end{aligned}$$

Since $y'' < 0$ for all values of x, the graph is always concave-down.

26 a

$$y = x^2 \ln x$$
$$y' = 2x \ln x + x$$
$$y'' = 2 \ln x + 3$$

If $y'' = 1$ then $\ln x = -1$ so $x = \mathrm{e}^{-1}$

$y(\mathrm{e}^{-1}) = -\mathrm{e}^{-2}$

The point is $(\mathrm{e}^{-1}, -\mathrm{e}^{-2})$

b At a point of inflection, $y'' = 0$.

$2 \ln x + 3 = 0$

$$x = \mathrm{e}^{-\frac{3}{2}}$$

27 a

$$y = x^3 - 5x^2 + 4x - 2$$
$$y' = 3x^2 - 10x + 4$$
$$y'' = 6x - 10$$

The gradient of the graph is decreasing where $y'' < 0$

$6x - 10 < 0$

$$x < \frac{5}{3}$$

b The point of inflection occurs when $y'' = 0$

$$y\left(\frac{5}{3}\right) = \frac{125}{27} - \frac{125}{9} + \frac{20}{3} - 2 = -\frac{124}{27}$$

The point of inflection is at $\left(\frac{5}{3}, -\frac{124}{27}\right)$

28 a

$$\mathrm{f}(x) = \frac{\ln x}{x} = x^{-1} \ln x$$
$$\mathrm{f}'(x) = -x^{-2} \ln x + x^{-2} = \frac{1 - \ln x}{x^2}$$

b Stationary point occurs when $\mathrm{f}'(x) = 0$

$1 - \ln x = 0$

$$x = \mathrm{e}$$

$\mathrm{f}(\mathrm{e}) = \mathrm{e}^{-1}$

Stationary point is $(\mathrm{e}, \mathrm{e}^{-1})$

c

$$\mathrm{f}'(x) = \frac{1 - \ln x}{x^2} = \frac{u}{v}$$
$$u = 1 - \ln x \text{ so } u' = -x^{-1}$$
$$v = x^2 \text{ so } v' = 2x$$
$$\mathrm{f}''(x) = \frac{u'v - uv'}{v^2} = \frac{-x - 2x(1 - \ln x)}{x^4} = \frac{2x \ln x - 3x}{x^4} = \frac{2 \ln x - 3}{x^3}$$

28 d

$$f''(e) = \frac{2-3}{e^3} = -e^{-3} < 0$$

So the stationary point is a local maximum.

29

$$y = x^{-1} + p^2 x$$
$$y' = -x^{-2} + p^2$$

Stationary point occurs when $y' = 0$ so $x = \pm p^{-1}$

$$y(\pm p^{-1}) = \pm(2p)$$
$$y'' = 2x^{-3}$$

$y''(p^{-1}) = 2p^3 > 0$ so $(p^{-1}, 2p)$ is a local minimum.

$y''(-p^{-1}) = -2p^3 < 0$ so $(-p^{-1}, -2p)$ is a local maximum.

30

$$y = e^{\sin x}$$
$$y' = \cos x \; e^{\sin x}$$

Stationary point occurs when $y' = 0$ so $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$$y\left(\frac{\pi}{2}\right) = e, y\left(\frac{3\pi}{2}\right) = e^{-1}$$

$$y'' = (\cos^2 x - \sin x)e^{\sin x}$$

$y''\left(\frac{\pi}{2}\right) = -e < 0$ so $\left(\frac{\pi}{2}, e\right)$ is a local maximum.

$y''\left(\frac{3\pi}{2}\right) = e^{-1} > 0$ so $\left(\frac{3\pi}{2}, e^{-1}\right)$ is a local minimum.

31

$$y = \sin x - \frac{x}{2}$$
$$y' = \cos x - \frac{1}{2}$$

Stationary point occurs when $y' = 0$ so $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

$$y\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}, y\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6}$$

$$y'' = -\sin x$$

$y''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} < 0$ so $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$ is a local maximum.

$y''\left(\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2} > 0$ so $\left(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2} - \frac{5\pi}{6}\right)$ is a local minimum.

32

$$y = \frac{x^2}{1+x}$$
$$y' = \frac{2x(1+x) - x^2}{(1+x)^2} = \frac{x^2+2x}{(1+x)^2}$$

Stationary point occurs when $y' = 0$ so $x = 0$ or -2

$$y(0) = 0, y(-2) = -4$$

$$y' = 1 - \frac{1}{(1+x)^2} = 1 - (1+x)^{-2}$$
$$y'' = 2(1+x)^{-3}$$
$y''(0) = 2 > 0$ so $(0, 0)$is a local minimum.
$y''(-2) = -2 < 0$ so $(-2, -4)$ is a local maximum.

33 a

$$P = 16x^{1.5} - x^3 - 25$$
$$P' = 24x^{0.5} - 3x^2$$

Stationary point occurs when $P' = 0$

$$x^{1.5} = 8$$
$$x = 4$$
$$P'' = 12x^{-0.5} - 6x$$
$P''(4) = 6 - 24 < 0$ so $x = 4$ represents a local maximum.

Since $P(x)$ is continuous for $x > 0$, and there is only one stationary point, this local maximum must be the global maximum.

The business should produce 4000 items each month to maximise profit.

Tip: Alternatively, a substitution shows simply what the maximum is and that it is global:

Let $u = x^{1.5}$ so $P = 16u - u^2 - 25$
Then P has a single turning point, which is the global maximum.
$$\frac{dP}{du} = 16 - 2u$$

So this maximum is at $u = 8$, or $x = 4$

b $P|_{x=4} = 16(8) - 64 - 25 = 39$

The maximum monthly profit is \$39 000.

34

$$V = x^2(a - x)$$
$$V' = 2xa - 3x^2$$

Stationary value when $V' = 0$

$$x = 0 \text{ or } \frac{2a}{3}$$

V is a negative cubic with a repeated root at 0 and a root at $x = a$, so has a local minimum at $x = 0$ and a local maximum for some x between 0 and a.

$$V\left(\frac{2a}{3}\right) = \frac{4a^2}{9}\left(\frac{a}{3}\right) = \frac{4a^3}{9} \text{ cm}^3$$

35

$$y = xe^{px}$$
$$y' = (1 + px)e^{px}$$

Stationary point where $y' = 0$, so $x = -p^{-1}$

$$y'' = (2p + p^2x)e^{px}$$
$$y''(-p^{-1}) = pe^{-1}$$
$$y(-p^{-1}) = -(pe)^{-1}$$

Stationary point $(-p^{-1}, -(pe)^{-1})$ is a local maximum if $p < 0$ and is a local minimum if $p > 0$.

36 $V = t^3 - 3t^2 + 5t$

$$\frac{dV}{dt} = 3t^2 - 6t + 5 = 3(t-1)^2 + 2$$

$\frac{dV}{dt}$ is a positive quadratic with vertex at $(1, 2)$ so the barrel will be filling most slowly at $t = 1$ hour, when $\frac{dV}{dt} = 2$ litres per hour.

37

$$P = \frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2} = 1 - (1+t^2)^{-1}$$
$$\frac{dP}{dt} = 2t(1+t^2)^{-2}$$
$$\frac{d^2P}{dt^2} = 2(1+t^2)^{-2} - 8t^2(1+t^2)^{-3}$$
$$= \frac{2(1+t^2) - 8t^2}{(1+t^2)^3}$$
$$= \frac{2-6t^2}{(1+t^2)^3}$$

Stationary point in rate of growth occurs when the second derivative is zero.

$$2 - 6t^2 = 0$$
$$t = \frac{1}{\sqrt{3}}$$

The denominator of the second derivative is always positive, and the numerator is negative for $t > 3^{-0.5}$, positive for $t < 3^{-0.5}$, which indicates that the stationary point must be a local maximum.

$$P|_{t=\frac{1}{\sqrt{3}}} = \frac{\left(\frac{1}{3}\right)}{1+\frac{1}{3}} = \frac{1}{4}$$

When the population is 250, it is growing most quickly in this model.

38

$$\mathrm{f}(x) = \mathrm{e}^x - ax$$
$$\mathrm{f}'(x) = \mathrm{e}^x - a$$

Stationary point is at $x = \ln a$ (for $a > 0$; if $a \leq 0$ then there is no stationary point)

$\mathrm{f}(\ln a) = a - a\ln a$

$\mathrm{f}''(x) = \mathrm{e}^x > 0$ for all x, so any stationary point must be a minimum.

If $a = 0$ then the function is just $\mathrm{f}(x) = \mathrm{e}^x$, which can take any positive value.

If $a < 0$ then $\mathrm{f}(x) \to -\infty$ as $x \to -\infty$ and $\mathrm{f}(x) \to \infty$ as $x \to \infty$ so it can take any value.

The range of the function is therefore:

$$\begin{cases} \mathrm{f}(x) > a(1 - \ln a)(\text{if } a > 0) \\ \mathrm{f}(x) > 0(\text{if } a = 0) \\ f(x) \in \mathbb{R}(\text{if } a < 0) \end{cases}$$

39

$$y = \mathrm{e}^{2x} - 6\mathrm{e}^x + 4x + 8$$
$$y' = 2\mathrm{e}^{2x} - 6\mathrm{e}^x + 4 = 2(\mathrm{e}^x - 1)(\mathrm{e}^x - 2)$$

Stationary points where $y' = 0$: $x = 0$ or $\ln 2$

$$y(0) = 1 - 6 + 0 + 8 = 3$$
$$y(\ln 2) = 4 - 12 + 4\ln 2 + 8 = 4\ln 2 = \ln 16$$
$$y'' = 4\mathrm{e}^{2x} - 6\mathrm{e}^x$$
$y''(0) = 4 - 6 < 0$ so $(0, 3)$ is a local maximum.

$y''(\ln 2) = 16 - 12 > 0$ so $(\ln 2, \ln 16)$ is a local minimum.

40

$$h(x) = \mathrm{e}^x + \mathrm{e}^{-2x}$$
$$h'(x) = \mathrm{e}^x - 2\mathrm{e}^{-2x}$$

Stationary value when $h'(x) = 0$

$$\mathrm{e}^{-2x}(\mathrm{e}^{3x} - 2) = 0$$
$$x = \frac{1}{3}\ln 2$$

$h'(x) > 0$ for $x > \frac{1}{3}\ln 2$ and $h'(x) < 0$ for $x < \frac{1}{3}\ln 2$ so this is a local minimum.

$$h\left(\frac{1}{3}\ln 2\right) = 2^{\frac{1}{3}} + 2^{-\frac{2}{3}} = 2^{\frac{1}{3}}\left(1 + \frac{1}{2}\right) = 1.5\sqrt[3]{2} = 1.89 \text{ m}$$

41 a

$$y = x^3 + ax^2 + bx + 7$$
$$y' = 3x^2 + 2ax + b$$
$$y'' = 6x + 2a$$

The curve is concave up where $y'' > 0$

$6x + 2a > 0$ for $x > -\frac{a}{3}$

$-\frac{a}{3} = 2$ so $a = -6$

41 b If the curve is to be strictly increasing throughout, then $y' > 0$ for all x

$3x^2 - 12x + b > 0$

Positive quadratic is always positive if it has no roots; that is, if the discriminant $\Delta < 0$.

$$\begin{aligned}\Delta = 144 - 12b &< 0\\ b &> 12\end{aligned}$$

42 a

$$\begin{aligned}y &= ax^3 - bx^2\\ y' &= 3ax^2 - 2bx\end{aligned}$$

Curve is increasing where $y' > 0$

$$\begin{aligned}3ax^2 - 2bx &> 0\\ x(3ax - 2b) &> 0\end{aligned}$$

Positive quadratic is greater than zero outside the roots so $x < 0$ or $x > \dfrac{2b}{3a}$

$$\begin{aligned}\frac{2b}{3a} &= 4\\ \frac{b}{a} &= 6\end{aligned}$$

b $y'' = 6ax - 2b$

The curve is concave up where $y'' > 0$

$$\begin{aligned}6ax - 2b &> 0\\ x &> \frac{1}{3}\frac{b}{a}\\ x &> 2\end{aligned}$$

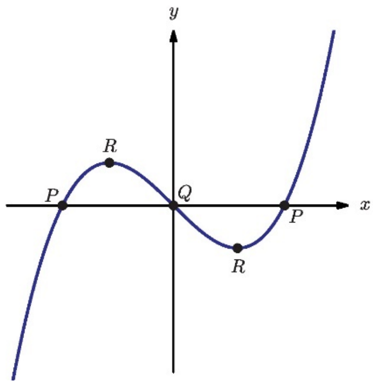
43

$$\begin{aligned}y''(x) &= 10\\ y'(x) &= 10x + c\\ y'(0) &= c = 10\\ y(x) &= 5x^2 + cx + d\\ y(0) &= d = 2\\ \text{So } y(x) &= 5x^2 + 10x + 2\end{aligned}$$

44 a Local minimum of $f(x)$ occurs when $f'(x) = 0$, with $f'(x) < 0$ to the left and $f'(x) > 0$ to the right.

b Local maximum of $f(x)$ occurs when $f'(x) = 0$, with $f'(x) > 0$ to the left and $f'(x) < 0$ to the right.

44 c Point of inflection of $f(x)$ occurs when $f'(x)$ has a stationary point which is not also an inflection. Stationary points where $f'(x)$ has an inflection would need further analysis, but there are no such points here.



d None of the points of inflection are stationary points, since none of the stationary points of $f'(x)$ occur at $f'(x) = 0$.

Mixed Practice

1

$$\begin{aligned} y &= x^2 + bx + c \\ y' &= 2x + b \\ y'(2) &= 0 = 4 + b \text{ so } b = -4 \\ y(2) &= 3 = 4 + 2b + c \text{ so } c = 7 \end{aligned}$$

2 $e^{x^2} = e^u$ where $u = x^2$

$$\frac{d}{dx}\left(e^{x^2}\right) = \frac{d}{du}(e^u) \times \frac{du}{dx} = e^u \times 2x = 2xe^{x^2}$$

3

$$\begin{aligned} y &= \frac{1}{x+2} = (x+2)^{-1} \\ y' &= -(x+2)^{-2} \\ y'' &= 2(x+2)^{-3} \end{aligned}$$

4

$\frac{\sin x}{x} = \frac{u}{v}$ where $u = \sin x, v = x$

Then $u' = \cos x, v' = 1$

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{u'v - uv'}{v^2} \\ &= \frac{x\cos x - \sin x}{x^2} \end{aligned}$$

5

$f(x) = x \ln x$

$f'(x) = \ln x + x \times \frac{1}{x} = 1 + \ln x$

6

$$\begin{aligned} y &= \sqrt{x-1} = (x-1)^{0.5} \\ y' &= 0.5(x-1)^{-0.5} \\ y'(5) &= \frac{0.5}{\sqrt{4}} = \frac{1}{4} \end{aligned}$$

7

$$\begin{aligned} y &= \ln x \\ y' &= \frac{1}{x} \\ y'(1) &= 1 \\ y(1) &= 0 \end{aligned}$$

Tangent at (1, 0) is $y = x - 1$

8

$$\begin{aligned} y &= \mathrm{e}^{2x} \\ y' &= 2e^{2x} \\ y'(0) &= 2 \end{aligned}$$

Normal gradient at (0, 1) is $-\frac{1}{2}$ so equation of normal is:

$$\begin{aligned} y - 1 &= -\frac{1}{2}(x-0) \\ y &= 1 - \frac{1}{2}x \end{aligned}$$

9 $V = x^3$

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} \\ \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{x=12} &= 3(12)^2 \times 0.6 = 259.2\ \mathrm{cm}^3\ \mathrm{s}^{-1} \end{aligned}$$

10

$$\begin{aligned} y &= 3\sin 2\pi x \\ y' &= 6\pi\cos 2\pi x \\ y'|_{x=\frac{7}{12}} &= 6\pi\cos\left(\frac{7\pi}{6}\right) = -3\pi\sqrt{3} = -16.3 \end{aligned}$$

11 a

$$\begin{aligned} a^2h &= 1000 \\ h &= \frac{1000}{a^2} \\ S &= 2a^2 + 4ah \\ &= 2a^2 + \frac{4000}{a} \end{aligned}$$

b

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}a} &= 4a - 4000a^{-2} = 0 \\ a^3 &= 1000 \\ a &= 10 \end{aligned}$$

11 c

$$\frac{\mathrm{d}^2S}{\mathrm{d}a^2} = 4 + 8000a^{-3}$$
$$\left.\frac{\mathrm{d}^2S}{\mathrm{d}a^2}\right|_{a=10} = 4 + 8 > 0 \text{ so } a = 10 \text{ represents a minimum of S.}$$

When $a = 10$, the cuboid is a cube of side 10 cm so $S = 600\ \mathrm{cm}^2$

12 $y = x\cos x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x - x\sin x$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 1$$
$$y|_{x=0} = 0$$

The tangent at $(0,0)$ is $y = x$

13 $y = x^2\mathrm{e}^{2x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (2x + 2x^2)\mathrm{e}^{2x}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (2 + 8x + 4x^2)\mathrm{e}^{2x}$$

14 a $A = \frac{1}{2}x(6 - x) = 3x - \frac{1}{2}x^2$

This is a negative quadratic so the stationary point is a maximum.

Minimum area is clearly at $x = 0$ or $x = 6$, from context.

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 3 - x$$

So maximum area is at $x = 3$.

$A(3) = 4.5\ \mathrm{cm}^2$

b

$$\begin{aligned} P &= x + (6 - x) + \sqrt{x^2 + (6 - x)^2} \\ &= 6 + (36 - 12x + 2x^2)^{0.5} \\ &= 6 + \sqrt{2(x - 3)^2 + 18} \end{aligned}$$

So $P - 6$ is the square root of a positive quadratic, hence P has a minimum.

From the completed square form above, or by differentiation, minimum P is at $x = 3$.

$$\frac{\mathrm{d}P}{\mathrm{d}x} = \frac{(2x - 6)}{\sqrt{36 - 12x + 2x^2}} \text{ so } P'(3) = 0$$
$$P(3) = 6 + \sqrt{18} = \left(6 + 3\sqrt{2}\right) \text{ cm}$$

15

$$\begin{aligned} \mathrm{f}(x) &= x\mathrm{e}^{kx} \\ \mathrm{f}'(x) &= (kx + 1)\mathrm{e}^{kx} \\ \mathrm{f}''(x) &= (k^2x + 2k)\mathrm{e}^{kx} \\ \mathrm{f}''(0) &= 10 = 2k \text{ so } k = 5 \\ \mathrm{f}'(1) &= 6\mathrm{e}^5 \end{aligned}$$

16 a

$$\begin{aligned} y &= x^3 - kx^2 + 8x + 2 \\ y' &= 3x^2 - 2kx + 8 \\ y'' &= 6x - 2k \end{aligned}$$

The graph is concave-up where $y'' > 0$, so for $x > \frac{k}{3}$

Then $\frac{k}{3} = 4$ so $k = 12$

b Inflection occurs where $y'' = 0$, which is at $x = 4$.

$y(4) = 64 - 16k + 32 + 2 = -94$

So the point of inflection is at $(4, -94)$

17

$$\begin{aligned} f(x) &= xe^x \sin x \\ f'(x) &= e^x \sin x + xe^x \sin x + xe^x \cos x \\ &= e^x\big((x+1)\sin x + x\cos x\big) \end{aligned}$$

18

$$\begin{aligned} f(x) &= x(4+x)^{-0.5} \\ f'(x) &= (4+x)^{-0.5} - 0.5x(4+x)^{-1.5} \\ &= (4 + x - 0.5x)(4+x)^{-1.5} \\ &= \frac{8+x}{2(4+x)^{1.5}} \end{aligned}$$

$a = 1, b = 8, c = 1.5$

19 a

$$\begin{aligned} f(x) &= xe^{-x} \\ f'(x) &= (1-x)e^{-x} \\ f''(x) &= (x-2)e^{-x} \end{aligned}$$

The curve is concave-down where $f''(x) < 0$

$x < 2$

b Inflection occurs where $y'' = 0$, which is at $x = 2$

$f(2) = 2e^{-2}$

So the point of inflection is at $(2, 2e^{-2})$

20

$$\begin{aligned} y &= e^{-x} \sin x \\ y' &= e^{-x}(\cos x - \sin x) \end{aligned}$$

Stationary point where $y' = 0$

$\cos x = \sin x \Rightarrow \tan x = 1$ so $x = \frac{\pi}{4}$

$y\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}$

The stationary point is $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}}\right)$

21 $z = xe^y$

$$\frac{dz}{dt} = \frac{dz}{dx} \times \frac{dx}{dt} + \frac{dz}{dy} \times \frac{dy}{dt}$$

$$= e^y \frac{dx}{dt} + xe^y \frac{dy}{dt}$$

When $x = 2, y = 1, \frac{dx}{dt} = -1$ and $\frac{dy}{dt} = 0.5$

$$\frac{dz}{dt} = e(-1) + 2e(0.5) = 0$$

22 a $V(0) = 5$

The initial volume is 5×10^6 m^3.

22 b

$$\frac{dV}{dt} = 2(1 - t)e^{-t}$$

Stationary value at $t = 1$

$$\frac{d^2V}{dt^2} = 2(t - 2)e^{-t}$$

$$\left.\frac{d^2V}{dt^2}\right|_{t=1} = -2e^{-1} < 1 \text{ so the stationary point is a local maximum.}$$

$$V(1) = 2e^{-1} + 5 \approx 5.74$$

Maximum volume is 5.74×10^6 m^3.

c Maximum rate of emptying when second derivative is zero.

$t = 2$

The lake is emptying fastest 2 hours after the storm.

23

$$C = \frac{2t}{3 + t^2} = \frac{u}{v}$$

$u = 2t$ so $u' = 2$

$v = 3 + t^2$ so $v' = 2t$

$$\frac{dC}{dt} = \frac{u'v - uv'}{v^2} = \frac{2(3 + t^2) - 4t^2}{(3 + t^2)^2} = \frac{6 - 2t^2}{(3 + t^2)^2}$$

Concentration is zero at $t = 0$, rises and then falls back towards zero as $t \to \infty$.

Maximum concentration occurs when $\frac{dC}{dt} = 0$.

$$6 - 2t^2 = 0$$

$$t = \sqrt{3}$$

$$C(\sqrt{3}) = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \approx 0.577 \text{ mg l}^{-1}$$

24 At time t hours, the positions of the two cyclists are $(0, 20t)$ and $(-15t, 0)$so the distance d between them is given by $d = \sqrt{(20t)^2 + (15t)^2} = t\sqrt{625} = 25t$

Then $\frac{\mathrm{d}d}{\mathrm{d}t} = 25$

That is, the distance between the cyclists increases at a constant rate of 25 km hour^{-1}.

25 a

$$\begin{aligned}
\mathrm{f}(x) &= \frac{x}{-2x^2 + 5x - 2} = \frac{u}{v} \\
u &= x \text{ so } u' = 1 \\
v &= -2x^2 + 5x - 2 \text{ so } v' = 5 - 4x \\
\mathrm{f}'(x) &= \frac{u'v - uv'}{v^2} \\
&= \frac{-2x^2 + 5x - 2 - x(5 - 4x)}{(-2x^2 + 5x - 2)^2} \\
&= \frac{2x^2 - 2}{(-2x^2 + 5x - 2)^2}
\end{aligned}$$

b Stationary points will occur when $\mathrm{f}'(x) = 0$

$$\begin{aligned}
x^2 &= 1 \\
x &= \pm 1
\end{aligned}$$

Since $A(1, 1))$ is known to be a local minimum, the local maximum must be at $x = -1$.

$$\mathrm{f}(-1) = \frac{-1}{-2 - 5 - 2} = \frac{1}{9}$$

B has coordinates $\left(-1, \frac{1}{9}\right)$

c If $y = k$ does not intersect the graph then:

$$\frac{1}{9} < k < 1$$

26

$$\begin{aligned}
y &= \ln x + kx \\
y' &= \frac{1}{x} + k \\
y'(1) &= 1 + k \\
y(1) &= k
\end{aligned}$$

Tangent at $(1, k)$ has equation:

$$\begin{aligned}
y - k &= (1 + k)(x - 1) \\
y &= (1 + k)x - 1
\end{aligned}$$

So the y-intercept of the tangent passes through $(0, -1)$ for all values of k.

27 a

$$\mathrm{f}(0) = \frac{100}{1 + 50} = \frac{100}{51}$$

27 b

$$\frac{100}{1+50\mathrm{e}^{-0.2x}} = 95$$
$$1+50\mathrm{e}^{-0.2x} = \frac{100}{95}$$
$$\mathrm{e}^{-0.2x} = \frac{1}{950}$$
$$x = 5\ln 950 = 34.3$$

c As $x \to \infty, \mathrm{f}(x) \to 100$

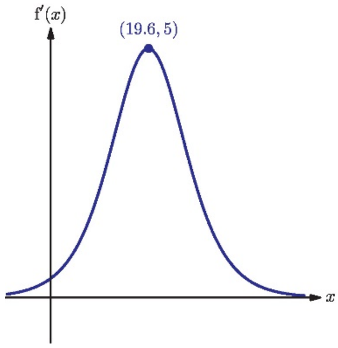
As $x \to -\infty, \mathrm{f}(x) \to 0$

The range is $0 < \mathrm{f}(x) < 100$.

d

$$\mathrm{f}(x) = \frac{100}{(1+50\mathrm{e}^{-0.2x})} = \frac{u}{v}$$
$u = 100$ so $u' = 0$
$v = 1+50\mathrm{e}^{-0.2x}$ so $v' = -10\mathrm{e}^{-0.2x}$
$$\mathrm{f}'(x) = \frac{u'v - uv'}{v^2}$$
$$= \frac{1000\mathrm{e}^{-0.2x}}{(1+50\mathrm{e}^{-0.2x})^2}$$

e From calculator, the graph of $\mathrm{f}'(x)$ has a bell-shape.



The maximum point is $(19.6, 5)$.
The maximum rate of change is 5.

28 a

$$\mathrm{f}(x) = \frac{x-a}{x-b} = \frac{u}{v}$$
$u = x - a$ so $u' = 1$
$v = x - b$ so $v' = 1$
$$\mathrm{f}'(x) = \frac{u'v - uv'}{v^2} = \frac{x-b-x+a}{(x-b)^2} = \frac{a-b}{(x-b)^2}$$

The denominator of $f'(x)$ is always positive for $x \neq b$.

If $a > b$ then $\mathrm{f}'(x) > 0$ for $x \neq b$.

b The graph has a discontinuity at $x = b$, so the inference $p > q \Rightarrow \mathrm{f}(p) > \mathrm{f}(q)$ only applies if both p and q are either both greater or both less than b.
If $q < b < p$ then the inference cannot be made.

29 a Require the argument of the square root to be non-negative for a real function.

$$4 - x^2 \geq 0$$
$$|x| \leq 2$$
$$-2 \leq x \leq 2$$

b

$$y = (4 - x^2)^{0.5}$$
$$y' = -x(4 - x^2)^{-0.5}$$

So at point $\left(a, \sqrt{4 - a^2}\right)$ the gradient is $-\dfrac{a}{\sqrt{4 - a^2}}$

and the normal gradient is $\dfrac{\sqrt{4 - a^2}}{a}$

The normal equation is:

$$y - \sqrt{4 - a^2} = \frac{\sqrt{4 - a^2}}{a}(x - a)$$
$$y = \frac{\sqrt{4 - a^2}}{a}x$$

For all a in the domain, the normal line passes through the origin.

30 a

$$y = kx^n$$
$$y' = nk$$

Then $kx^n \times nkx^{n-1} = 1$

$$nk^2x^{2n-1} = 1$$

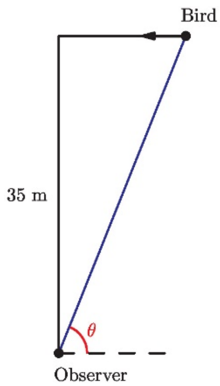
for all x, so

$$2n - 1 = 0$$
$$n = \frac{1}{2}$$

b

$$\frac{1}{2}k^2 = 1$$
$$k = \pm\sqrt{2}$$

31 If d is the distance between observer and bird, then:



$$d = \frac{35}{\sin\theta} = 35\,\mathrm{cosec}\,\theta$$

$$\frac{\mathrm{d}d}{\mathrm{d}t} = -35\,\mathrm{cosec}\,\theta\cot\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

When $\theta = 1.2, \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{60}$

$$\frac{\mathrm{d}d}{\mathrm{d}t} = -35\,\mathrm{cosec}\,1.2\cot 1.2 \times \frac{1}{60} = 0.243\text{ m s}^{-1}$$

11 Integration

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 11A

23

$$\begin{aligned} y &= \int 3x^{0.5}\ \mathrm{d}x \\ &= 2x^{1.5} + c \\ y(9) &= 12 = 2 \times 27 + c \\ c &= -42 \\ y &= 2x^{1.5} - 42 \end{aligned}$$

24

$$\begin{aligned} \mathrm{f}(x) &= \int 2\cos x - 3\sin x\ \mathrm{d}x \\ &= 2\sin x + 3\cos x + c \\ \mathrm{f}(0) &= 5 = 3 + c \\ c &= 2 \\ \mathrm{f}(x) &= 2\sin x + 3\cos x + 2 \end{aligned}$$

25

$$\begin{aligned} y &= \int 2\mathrm{e}^{x} - 5x^{-1}\ \mathrm{d}x \\ &= 2\mathrm{e}^{x} - 5\ln x + c \\ y(1) &= 0 = 2\mathrm{e} + c \\ c &= -2\mathrm{e} \\ y &= 2(\mathrm{e}^{x} - \mathrm{e}) - 5\ln|x| \end{aligned}$$

26

$$\int 2x + 1.5x^{-1}\,\mathrm{d}x = x^2 + 1.5\ln|x| + c$$

27

$$\begin{aligned} V &= \int t - 0.5\sin t\ \mathrm{d}t \\ &= 0.5t^2 + 0.5\cos t + c \\ V(0) &= 2 = 0.5 + c \\ c &= 1.5 \\ V &= 0.5(t^2 + \cos t + 3) \end{aligned}$$

28

$$\begin{aligned} x &= \int 5 - 2e^t \, dt \\ &= 5t - 2e^t + c \\ x(0) &= 5 = c - 2 \\ c &= 7 \\ x &= 5t - 2e^t + 7 \end{aligned}$$

29 If $g(x) = 5 - 2x$ then $g'(x) = -2$

$$\begin{aligned} \int (5-2x)^6 \, dx &= -\frac{1}{2}\int \left(g(x)\right)^6 g'(x) \, dx \\ &= -\frac{1}{2}\left(\frac{1}{7}\left(g(x)\right)^7\right) + c \\ &= c - \frac{1}{14}(5-2x)^7 \end{aligned}$$

30

$$\int 3\sin 2x - 2\cos 3x \, dx = -\frac{3}{2}\cos 2x - \frac{2}{3}\sin 3x + c$$

31 If $g(x) = x^2 + 1$ then $g'(x) = 2x$

$$\begin{aligned} \int \left(g(x)\right)^5 g'(x) \, dx &= \frac{1}{6}\left(g(x)\right)^6 \\ &= \frac{1}{6}(x^2+1)^6 + c \end{aligned}$$

32 If $g(x) = 3x^2$ then $g'(x) = 6x$

$$\begin{aligned} y' &= \frac{1}{6}g'(x)\sin\left(g(x)\right) \\ y &= -\frac{1}{6}\cos\left(g(x)\right) + c \\ &= c - \frac{1}{6}\cos(3x^2) \end{aligned}$$

33 If $g(x) = x^2 + 2$ then $g'(x) = 2x$

$$\begin{aligned} \int \frac{3x}{\sqrt{x^2+2}} \, dx &= \int 1.5\, g'(x)\left(g(x)\right)^{-0.5} \, dx \\ &= 3\left(g(x)\right)^{0.5} + c \\ &= 3\sqrt{x^2+2} + c \end{aligned}$$

34

$$\begin{aligned} f(x) &= \int (2x+3)^{-2} \, dx \\ &= -\left(\frac{1}{2}\right)(2x+3)^{-1} + c \\ &= c - \frac{1}{4x+6} \\ f(-1) &= 1 = c - \frac{1}{2} \\ c &= \frac{3}{2} \end{aligned}$$

35 Let $\mathrm{g}(x) = \ln x$ then $\mathrm{g}'(x) = x^{-1}$

$$f' = \frac{2}{3}\mathrm{g}'(x)\mathrm{g}(x)$$
$$f = \frac{1}{3}\big(\mathrm{g}(x)\big)^2 + c$$
$$= \frac{1}{3}(\ln x)^2 + c$$

$\mathrm{f}(1) = 1 = c$

$$\mathrm{f}(x) = 1 + \frac{(\ln x)^2}{3}$$

36

$$\mathrm{f}(x) = \int 2x^{-1}\ \mathrm{d}x$$
$$= 2\ln|x| + c$$

$\mathrm{f}(-1) = 5 = c$

$\mathrm{f}(-3) = 5 + 2\ln 3$

37

$$\int 4\,(\cos x)^3 \sin x + kx^2 + 1 = -\cos^4 x + \frac{k}{3}x^3 + x + c$$

38

$$\int \mathrm{e}^x - \mathrm{e}^{-3x}\,\mathrm{d}x = \mathrm{e}^x + \frac{1}{3}\,\mathrm{e}^{-3x} + c$$

39 Let $\mathrm{g}(x) = x^2 + 1$ then $\mathrm{g}'(x) = 2x$

$$\int x\sqrt{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int \big(\mathrm{g}(x)\big)^{0.5} g'(x)\,\mathrm{d}x$$
$$= \frac{1}{3}\big(\mathrm{g}(x)\big)^{1.5} + c$$
$$= \frac{1}{3}(x^2+1)^{1.5} + c$$

40 **a** $\cos 2x = 1 - 2\sin^2 x$

b

$$\int \sin^2 x\ \mathrm{d}x = \int \frac{1}{2}(1 - \cos 2x)\ \mathrm{d}x$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

41 Let $\mathrm{g}(x) = \cos x$ then $\mathrm{g}'(x) = -\sin x$

$$\int \tan x\ \mathrm{d}x = -\int \frac{\mathrm{g}'(x)}{\mathrm{g}(x)}\ \mathrm{d}x$$
$$= -\ln|\mathrm{g}(x)| + c$$
$$= c - \ln|\cos x|$$

Worked solutions

42

$$\int \sin x\,(1-\cos^2 x)\ \mathrm{d}x = -\cos x + \frac{1}{3}\cos^3 x + c$$

43 Let $\mathrm{g}(x) = \ln x$ then $\mathrm{g}'(x) = x^{-1}$

$$\begin{aligned}\int \frac{1}{x\ln x}\ \mathrm{d}x &= \int \frac{\mathrm{g}'(x)}{\mathrm{g}(x)}\ \mathrm{d}x\\ &= \ln|\mathrm{g}(x)| + c\\ &= \ln|\ln x| + c\end{aligned}$$

Exercise 11B

13

$$\begin{aligned}\int_1^5 (3x+1)^{0.5}\ \mathrm{d}x &= \left[\frac{1}{3}\left(\frac{1}{1.5}\right)(3x+1)^{1.5}\right]_1^5\\ &= \frac{16^{1.5}-4^{1.5}}{4.5}\\ &= \frac{112}{9}\end{aligned}$$

14

$$\begin{aligned}\int_0^{\frac{\pi}{2}} 3\sin 2x\ \mathrm{d}x &= \left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{2}}\\ &= \frac{3}{2}+\frac{3}{2} = 3\end{aligned}$$

15

$$\int_2^5 \sqrt{\ln x}\ \mathrm{d}x = 3.29\ \text{(GDC)}$$

16

$$\begin{aligned}\int_1^a 3x^{-0.5}\ \mathrm{d}x &= \left[6\sqrt{x}\right]_1^a\\ &= 6\left(\sqrt{a}-1\right) = 24\\ \sqrt{a}-1 &= 4\\ a &= 25\end{aligned}$$

17 **a** $y = (3-x)(3+x)$ has roots $(\pm 3, 0)$

b

$$\begin{aligned}\int_{-3}^3 9-x^2\ \mathrm{d}x &= \left[9x - \frac{1}{3}x^3\right]_{-3}^3\\ &= (27-9)-(-27+9)\\ &= 36\end{aligned}$$

18 a

$$\int_{-1}^{1} x^3 - 2x^2 - x + 2 \ \mathrm{d}x = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-1}^{1}$$
$$= \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right) - \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2\right)$$
$$= \frac{8}{3}$$

$$\int_{1}^{2} x^3 - 2x^2 - x + 2 \ \mathrm{d}x = \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{1}^{2}$$
$$= \left(4 - \frac{16}{3} - 2 + 4\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right)$$
$$= -\frac{5}{12}$$

b Total area enclosed:

$$\frac{8}{3} + \frac{5}{12} = \frac{37}{12}$$

19

$$\int_{-3}^{3} \ln(10 - x^2) \ \mathrm{d}x = 11.0 \text{ (GDC)}$$

Tip: Solving exactly without a GDC:

$$\int_{-3}^{3} \ln(10 - x^2) \ \mathrm{d}x = \int_{-3}^{3} \ln\left((\sqrt{10} + x)(\sqrt{10} - x)\right) \ \mathrm{d}x$$
$$= \int_{-3}^{3} \ln(\sqrt{10} + x) + \ln(\sqrt{10} - x) \ \mathrm{d}x$$

Using $\int \ln x \ \mathrm{d}x = x \ln x - x + c$:

$$\int_{-3}^{3} \ln(\sqrt{10} + x) + \ln(\sqrt{10} - x) \ \mathrm{d}x$$
$$= \left[(\sqrt{10} + x)\ln(\sqrt{10} + x) - (\sqrt{10} + x) - (\sqrt{10} - x)\ln(\sqrt{10} - x) + (\sqrt{10} - x)\right]_{-3}^{3}$$
$$= \left[\sqrt{10}\left(\ln(\sqrt{10} + x) - \ln(\sqrt{10} - x)\right) + x\left(\ln(\sqrt{10} + x) + \ln(\sqrt{10} - x)\right) - 2x\right]_{-3}^{3}$$
$$= \left[\sqrt{10}\ln\left(\frac{\sqrt{10} + x}{\sqrt{10} - x}\right) + x\ln(10 - x^2) - 2x\right]_{-3}^{3}$$
$$= \sqrt{10}\ln\left(\frac{\sqrt{10} + 3}{\sqrt{10} - 3}\right) - \sqrt{10}\ln\left(\frac{\sqrt{10} - 3}{\sqrt{10} + 3}\right) - 12$$
$$= 2\sqrt{10}\ln\left(\frac{\sqrt{10} + 3}{\sqrt{10} - 3}\right) - 12$$

20

$$\int_0^{\ln 3} 2e^{-3x}\,dx = \left[-\frac{2}{3}e^{-3x}\right]_0^{\ln 3}$$
$$= \frac{2}{3}\left(1 - e^{-3\ln 3}\right)$$
$$= \frac{2}{3}\left(1 - \frac{1}{27}\right)$$
$$= \frac{52}{81}$$

21

$$\int_{-9}^{-3} 5x^{-1}\,dx = [5\ln|x|]_{-9}^{-3}$$
$$= 5(\ln 3 - \ln 9)$$
$$= -5\ln 3$$

22

$$\int_k^{2k} x^{-1}\,dx = [\ln x]_k^{2k}$$
$$= \ln 2k - \ln k$$
$$= \ln\left(\frac{2k}{k}\right)$$
$$= \ln 2$$

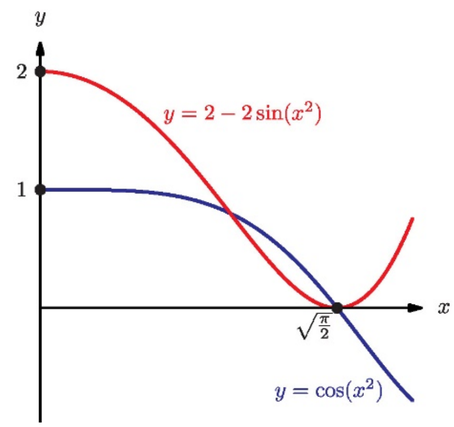
23 a By symmetry, the graphs intersect at $x = \frac{\pi}{4}$

So point of intersection is $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$.

b Again using symmetry:

$$\text{Shaded area} = 2\int_0^{\frac{\pi}{4}} \sin x\ dx$$
$$= 2[-\cos x]_0^{\frac{\pi}{4}}$$
$$= 2\left(1 - \frac{\sqrt{2}}{2}\right)$$
$$= 2 - \sqrt{2}$$

24 a



24 b Intersections where $\cos x^2 = 2 - 2\sin x^2$

From GDC: $x = \cos^{-1}(0.8) = 0.802$ or $\sqrt{\frac{\pi}{2}} = 1.25$

$$\int_{0.802}^{1.25} (2 - 2\sin x^2 - \cos x^2)\ \mathrm{d}x = 0.0701$$

25 The graph of $y = \sin 2x$ crosses the x-axis at $\frac{\pi}{2}$ so this question needs to be solved in two parts.

$$\int_0^{\frac{\pi}{2}} \sin 2x\,\mathrm{d}x = 1$$

$$\int_{\frac{\pi}{2}}^{\pi} \sin 2x\,\mathrm{d}x = -1$$

Total enclosed area $= 1 + 1 = 2$

26 Let $\mathrm{g}(x) = x^2 + 1$ so $\mathrm{g}'(x) = 2x$

$$\begin{aligned}\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,\mathrm{d}x &= \frac{1}{2}\int_0^{\sqrt{3}} \sqrt{\mathrm{g}(x)}\,\mathrm{g}'(x)\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{2}{3}\left(\mathrm{g}(x)\right)^{1.5}\right]_0^{\sqrt{3}} \\ &= \left[\frac{1}{3}(x^2+1)^{1.5}\right]_0^{\sqrt{3}} \\ &= \frac{8}{3} - \frac{1}{3} \\ &= \frac{7}{3}\end{aligned}$$

27

$$\begin{aligned}\text{Shaded area} &= \int_{-2}^{0} \mathrm{e}^x\ \mathrm{d}x + \int_0^1 \mathrm{e}^{-2x}\ \mathrm{d}x \\ &= [\mathrm{e}^x]_{-2}^{0} + \left[-\frac{1}{2}\mathrm{e}^{-2x}\right]_0^1 \\ &= 1 - \mathrm{e}^{-2} - \frac{1}{2}\mathrm{e}^{-2} + \frac{1}{2} \\ &= \frac{3(1-\mathrm{e}^{-2})}{2}\end{aligned}$$

28

$$\begin{aligned}\int_1^4 (2x+3)^{-1}\,\mathrm{d}x &= \left[\frac{1}{2}\ln|2x+3|\right]_1^4 \\ &= \frac{1}{2}(\ln 11 - \ln 5) \\ &= \frac{1}{2}\ln\left(\frac{11}{5}\right)\end{aligned}$$

29

$$\int_2^3 (x-5)^{-1}\,\mathrm{d}x = [\ln|x-5|]_2^3$$
$$= \ln 2 - \ln 3$$
$$= \ln\left(\frac{2}{3}\right)$$

30

$$\int_0^a x^2 + x\ \mathrm{d}x = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_0^a$$
$$= \frac{1}{6}(2a^3 + 3a^2) = 90$$
$$2a^3 + 3a^2 - 540 = 0$$
$$(a-6)(2a^2 + 9a + 90) = 0$$

From GDC, the only real solution is $a = 6$.

31

$$\int_a^{2a} x + 1\ \mathrm{d}x = \left[\frac{1}{2}x^2 + x\right]_a^{2a}$$
$$= \frac{1}{2}(4a^2 + 4a - a^2 - a) = 8$$
$$3a^2 + 3a - 16 = 0$$
$$(3a+8)(a-2) = 0$$
$$a = 2 \text{ or } -\frac{8}{3}$$

32 a At the intersections, $\mathrm{e}^{-x^2} = e^{-x}$

Taking logarithms: $x^2 = x$ so $x = 0$ or 1

A: $(0, 1), B(1,\ \mathrm{e}^{-1})$

b

$$\int_0^1 \left(\mathrm{e}^{-x^2} - \mathrm{e}^{-x}\right)\ \mathrm{d}x = 0.115 \text{ (GDC)}$$

33

$$\int_2^5 (3\mathrm{f}(x) - 1)\,\mathrm{d}x = 3\int_2^5 \mathrm{f}(x)\,\mathrm{d}x - \int_2^5 1\,\mathrm{d}x$$
$$= 30 - [x]_2^5$$
$$= 30 - (5-2)$$
$$= 27$$

34 Let $u = 2x$ so $\mathrm{d}u = 2\mathrm{d}x$

When $x = 3, u = 6$ and when $x = 0, u = 0$

$$\int_{x=0}^{x=3} 5\mathrm{f}(2x)\,\mathrm{d}x = \frac{1}{2} \times 5\int_{u=0}^{u=6} \mathrm{f}(u)\ \mathrm{d}u$$
$$= \frac{5}{2}(7)$$
$$= 17.5$$

35 a

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\ln x) = \ln x + x\left(\frac{1}{x}\right) = 1 + \ln x$$

b

$$\begin{aligned}\int_1^{\mathrm{e}} \ln x \,\mathrm{d}x &= \int_1^{\mathrm{e}} 1 + \ln x - 1 \;\mathrm{d}x\\ &= [x\ln x - x]_1^{\mathrm{e}}\\ &= (\mathrm{e} - \mathrm{e}) - (0 - 1)\\ &= 1\end{aligned}$$

36

$$\begin{aligned}\int_a^d |f(x)|\,\mathrm{d}x = 17 &= \int_a^b f(x)\,\mathrm{d}x + \int_c^d f(x)\,\mathrm{d}x - \int_b^c f(x)\,\mathrm{d}x\\ &= \int_a^d f(x)\,\mathrm{d}x - 2\int_b^c f(x)\,\mathrm{d}x\\ &= 5 - 2\int_b^c f(x)\,\mathrm{d}x\end{aligned}$$

$$\Rightarrow \int_b^c f(x)\,\mathrm{d}x = -6$$

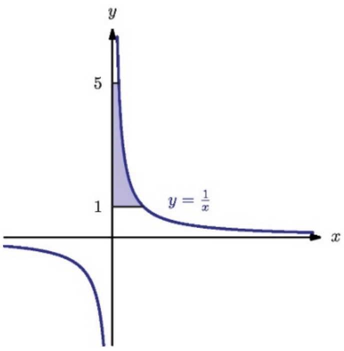
$$\begin{aligned}\int_a^d f(x)\,\mathrm{d}x = 5 &= \int_a^b f(x)\,\mathrm{d}x + \int_b^d f(x)\,\mathrm{d}x\\ &= \int_a^b f(x)\,\mathrm{d}x + 1\end{aligned}$$

$$\Rightarrow \int_a^b f(x)\,\mathrm{d}x = 4$$

$$\int_a^c \mathrm{f}(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = -2$$

Exercise 11C

13



a

$$y = \frac{1}{x} \text{ so } x = \frac{1}{y}$$

$$\int_{0.2}^{1} y^{-1}\;\mathrm{d}y = [\ln x]_{0.2}^{1} = 0 - \ln\left(\frac{1}{5}\right) = \ln 5$$

13 b

$$\pi\int_{0.2}^{1}(y^{-1})^2\ \mathrm{d}y = 4\pi$$

14

$$\pi\int_{1}^{a}(x^{-1})^2\ \mathrm{d}x = \pi[-x^{-1}]_1^a$$
$$= \pi\left(1-\frac{1}{a}\right) = \frac{2}{3}\pi$$
$$a = 3$$

15 $y = x^2$ so $x = \sqrt{y}$

$$\int_{0}^{a^2} x^2\ \mathrm{d}y = \int_{0}^{a^2} y\ \mathrm{d}y = \left[\frac{1}{2}y^2\right]_0^{a^2}$$
$$= \frac{1}{2}a^4 = 8\pi$$
$$a = \pm 2$$

16 a x-coordinate of A is -3

b

$$V = \pi\int_{-3}^{3} y^2\ \mathrm{d}x$$
$$= \pi\int_{-3}^{3} x+3\ \mathrm{d}x$$
$$= 18\pi$$

17 a

$$A = \int_{-2}^{2} x\ \mathrm{d}y$$
$$= \int_{-2}^{2} 4-y^2\ \mathrm{d}y$$
$$= \frac{32}{3}$$

b

$$V = \pi\int_{-2}^{2} x^2\ \mathrm{d}y$$
$$= \pi\int_{-2}^{2} 16-8y^2+y^4\ \mathrm{d}y$$
$$= \frac{512\pi}{15}$$

18 a

$$A = \int_{0}^{\pi} y\ \mathrm{d}x$$
$$= \int_{0}^{\pi} \sqrt{x}\sin x\,\mathrm{d}x$$
$$= 2.43\ \text{(GDC)}$$

18 b

$$\begin{aligned} V &= \pi \int_0^{\pi} y^2 \ \mathrm{d}x \\ &= \pi \int_0^{\pi} x \sin^2 x \ \mathrm{d}x \\ &= 7.75 \text{ (GDC)} \end{aligned}$$

19 a

$$\begin{aligned} y &= \sqrt{x^3 + 9} \\ y^2 &= x^3 + 9 \\ x^3 &= y^2 - 9 \\ x &= \sqrt[3]{y^2 - 9} \\ x^2 &= \sqrt[3]{(y^2 - 9)^2} \end{aligned}$$

b

$$\begin{aligned} A &= \int_0^3 -x \ \mathrm{d}y \\ &= \int_0^3 -(y^2 - 9)^{\frac{1}{3}} \ \mathrm{d}y \\ &= 5.25 \text{ (GDC)} \end{aligned}$$

c i

$$\begin{aligned} V_x &= \pi \int_{-\sqrt[3]{9}}^{0} y^2 \ \mathrm{d}x \\ &= \pi \int_{-\sqrt[3]{9}}^{0} x^3 + 9 \ \mathrm{d}x \\ &= 44.1 \text{ (GDC)} \end{aligned}$$

ii

$$\begin{aligned} V_y &= \pi \int_0^3 x^2 \ \mathrm{d}y \\ &= \pi \int_0^3 \sqrt[3]{(y^2 - 9)^2} \ \mathrm{d}x \\ &= 30.1 \text{ (GDC)} \end{aligned}$$

20 Boundary points are $(1, 1)$ and $\left(3, \frac{1}{3}\right)$, and the curve is $x = \frac{1}{y}$

$$\begin{aligned} V &= \pi \int_{\frac{1}{3}}^{1} x^2 \ \mathrm{d}y \\ &= \pi \int_{\frac{1}{3}}^{1} y^{-2} \ \mathrm{d}y \\ &= \pi \left[-\frac{1}{y}\right]_{\frac{1}{3}}^{1} \\ &= \pi(3 - 1) \\ &= 2\pi \end{aligned}$$

21 The volume if the entire rectangle with opposite vertices at the origin and at $(20,\ 4)$ is $\pi \times 20^2 \times 4$.

The volume of revolution of the curve arc will give the inner volume V_y, so the required volume is the difference.

$y = \sqrt{x-4}$ so $x = 4 + y^2$

$$V_y = \pi \int_0^4 x^2\ \mathrm{d}y$$
$$= \pi \int_0^4 16 + 8y^2 + y^4\ \mathrm{d}y$$
$$= \frac{6592}{15}\pi$$

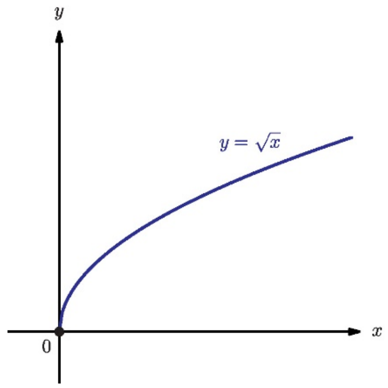
Then the required volume V is given by:

$$V = \left(1600 - \frac{6592}{15}\right)\pi = \frac{17408}{15}\pi \approx 3646$$

22

$$V_x = \pi \int_0^\pi y^2\ \mathrm{d}x$$
$$= \pi \int_0^\pi \sin^2 x\ \mathrm{d}x$$
$$= \frac{\pi^2}{2}$$

23 **a**



b

$$V_x = \pi \int_0^9 y^2\ \mathrm{d}x$$
$$= \pi \int_0^9 x\ \mathrm{d}x$$
$$= \frac{81}{2}\pi \approx 127$$

23 c $y = \sqrt{x}$ so $x = y^2$

The limits of the curve arc are $(0, 0)$ and $(9, 3)$

$$V_y = \pi \int_0^3 x^2 \, \mathrm{d}y$$
$$= \pi \int_0^3 y^4 \, \mathrm{d}y$$
$$= \frac{243}{5}\pi \approx 153$$

24 a $A: (0, 2), B: (2, 0)$

b

$$\text{Area} = \int_0^2 y \, \mathrm{d}x$$
$$= \int_0^2 (2 - x)\mathrm{e}^x \, \mathrm{d}x$$
$$= 4.39 \text{ (GDC)}$$

c

$$V_x = \pi \int_0^2 y^2 \, \mathrm{d}x$$
$$= \pi \int_0^2 (2 - x)^2 \mathrm{e}^{2x} \, \mathrm{d}x$$
$$= 32.7 \text{ (GDC)}$$

25 y-intercept is at $(0, 1)$

$y = \sqrt{x^2 + 1}$ so $x^2 = y^2 - 1$

$$V_y = \pi \int_1^3 x^2 \, \mathrm{d}y$$
$$= \pi \int_1^3 y^2 - 1 \, \mathrm{d}y$$
$$= \pi \left[\frac{1}{3}y^3 - y\right]_1^3$$
$$= \frac{20}{3}\pi$$

26 Volume of revolution of the shaded region will be given by the difference between the volume of revolution of the curve arc V_y.
and volume of revolution of the line (the latter producing a cylinder).
Intersection points are (2, 0) and (2, 2).

$$V = \pi \int_0^2 x^2 \, \mathrm{d}y - \pi \times 2^2 \times 2$$
$$= \pi \int_0^2 (2 + 2y - y^2)^2 \, \mathrm{d}y - 8\pi$$
$$= \frac{72}{5}\pi - 8\pi \text{ (GDC)}$$
$$= \frac{32}{5}\pi$$

27 a

$$A = \int_0^{\ln a} x \ \mathrm{d}y$$
$$= \int_0^{\ln a} \mathrm{e}^y \ \mathrm{d}y$$
$$= [\mathrm{e}^y]_0^{\ln a}$$
$$= a - 1$$

b The shaded area can also be seen as the area of the rectangle formed by the axes and with vertices at the origin and $(a, \ln a)$ less the unshaded area under the curve between $(1, 0)$ and $(a, \ln a)$.

$$a - 1 = a \ln a - \int_1^a y \ \mathrm{d}x$$
$$= a \ln a - \int_1^a \ln x \ \mathrm{d}x$$

So $\displaystyle\int_1^a \ln x \ \mathrm{d}x = a \ln a - a + 1$

28 a Gradient is $-\frac{h}{r}$, with intercept $(0, h)$

Line has equation $y = -\dfrac{h}{r}x + h$

$$hx + ry = rh$$

b If the line between the two points is rotated about the y-axis, the resultant shape is a cone of radius r and height h, with axis along the y-axis.

$$V = \pi \int_0^h x^2 \ \mathrm{d}y$$
$$= \pi \int_0^h \left(\frac{rh - ry}{h}\right)^2 \mathrm{d}y$$
$$= \pi \int_0^h \frac{r^2}{h^2}(h^2 - 2hy + y^2) \ \mathrm{d}y$$
$$= \frac{\pi r^2}{h^2}\left[h^2 y - hy^2 + \frac{1}{3}y^3\right]_0^h$$
$$= \frac{\pi r^2}{h^2}\left(h^3 - h^3 + \frac{1}{3}h^3\right)$$
$$= \frac{1}{3}\pi r^2 h$$

29 a $x^2 + y^2 = r^2$

29 b Considering the curve $y = \sqrt{r^2 - x^2}$ for x between $-r$ and r, a sphere is formed by rotating the curve about the x-axis:

$$\begin{aligned}
V &= \pi \int_{-r}^{r} y^2 \ \mathrm{d}x \\
&= \pi \int_{-r}^{r} r^2 - x^2 \ \mathrm{d}x \\
&= \pi \left[r^2 x - \frac{1}{3}x^3 \right]_{-r}^{r} \\
&= \pi \left(r^3 - \frac{1}{3}r^3 + r^3 - \frac{1}{3}r^3 \right) \\
&= \frac{4}{3}\pi r^3
\end{aligned}$$

30 a $y_1 = x^2, y_2 = \sqrt{x}$

Intersections where $y_1 = y_2$

$$\begin{aligned}
x^2 &= \sqrt{x} \\
x^4 &= x \\
x^3 &= 1 \text{ or } 0 \\
x &= 1 \text{ or } 0
\end{aligned}$$

Intersections are at $(0, 0)$ and $(1, 1)$

b

$$\begin{aligned}
V_x &= \pi \int_{0}^{1} y_2^2 - y_1^2 \ \mathrm{d}x \\
&= \pi \int_{0}^{1} x - x^4 \ \mathrm{d}x \\
&= \frac{3}{10}\pi
\end{aligned}$$

31 The region is symmetrical about $x = \frac{\pi}{4}$ so the calculation can be made by just considering the part of the region under $y = \sin x$ and doubling the result.

$$\begin{aligned}
V &= 2\pi \int_{0}^{\frac{\pi}{4}} \sin^2 x \ \mathrm{d}x \\
&= 2\pi \int_{0}^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos 2x) \ \mathrm{d}x \\
&= 2\pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x \right]_{0}^{\frac{\pi}{4}} \\
&= 2\pi \left(\frac{\pi}{8} - \frac{1}{4} \right) \\
&= \frac{\pi^2 - 2\pi}{4}
\end{aligned}$$

32 a $y_1 = x^2, y_2 = 2x$

Intersections where $y_1 = y_2$

$x^2 = 2x$

$x = 0$ or 2

Intersections are $(0, 0)$ and $(2, 4)$.

b

$$V_y = \pi \int_0^4 x_1^2 - x_2^2 \ \mathrm{d}y$$

$$= \pi \int_0^4 y - \left(\frac{y}{2}\right)^2 \ \mathrm{d}y$$

$$= \frac{8}{3}\pi$$

33 The boundary points on the curve are at (p, p^3) and (q, q^3).

$$A = \int_p^q y \ \mathrm{d}x = \left[\frac{1}{4}x^4\right]_p^q = \frac{1}{4}(q^4 - p^4)$$

$$B = \int_{p^3}^{q^3} x \ \mathrm{d}y = \int_{p^3}^{q^3} y^{\frac{1}{3}} \ \mathrm{d}y = \left[\frac{3}{4}y^{\frac{4}{3}}\right]_{p^3}^{q^3} = \frac{3}{4}(q^4 - p^4)$$

The ratio of $A : B$ is therefore $1 : 3$, independent of p and q.

34 a $y = \ln(x - 2)$

b $x = \mathrm{e}^y + 2$

The volume obtained rotating $y = \ln x$ for $1 \leqslant x \leqslant \mathrm{e}$ is the same as when the curve $y = \ln(x - 2)$ for $3 \leqslant x \leqslant \mathrm{e} + 2$ is rotated about the line $x = 0$.
Boundary points are $(3, 0)$ and $(\mathrm{e} + 2, 1)$.

$$V_y = \pi \int_0^1 x^2 \, \mathrm{d}y$$

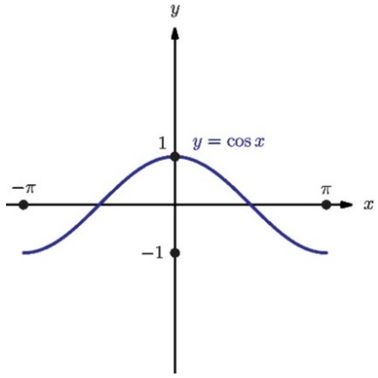
$$= \pi \int_0^1 \mathrm{e}^{2y} + 4\mathrm{e}^y + 4 \ \mathrm{d}y$$

$$= \pi \left[\frac{1}{2}\mathrm{e}^{2y} + 4\mathrm{e}^y + 4y\right]_0^1$$

$$= \pi \left(\frac{1}{2}\mathrm{e}^2 + 4\mathrm{e} + 4 - \frac{1}{2} - 4\right)$$

$$= \frac{\pi}{2}(\mathrm{e}^2 + 8\mathrm{e} - 1)$$

35 a



b The volume obtained is the same as for $y = \cos x + 1$ rotated about the line $y = 0$ (the x-axis).

$$\begin{aligned} V_x &= \pi \int_{-\pi}^{\pi} y^2 \ \mathrm{d}x \\ &= \pi \int_{-\pi}^{\pi} \cos^2 x + 2\cos x + 1 \ \mathrm{d}x \\ &= \pi \int_{-\pi}^{\pi} \frac{1}{2}(\cos 2x + 1) + 2\cos x + 1 \ \mathrm{d}x \\ &= \frac{\pi}{2} \int_{-\pi}^{\pi} \cos 2x + 4\cos x + 3 \ \mathrm{d}x \end{aligned}$$

c

$$\begin{aligned} V_x &= \pi \int_{-\pi}^{\pi} \frac{1}{2}(\cos 2x + 1) + 2\cos x + 1 \ \mathrm{d}x \\ &= \pi \left[\frac{1}{4}\sin 2x + 2\sin x + \frac{3}{2}x\right]_{-\pi}^{\pi} \\ &= 3\pi^2 \end{aligned}$$

Exercise 11D

11 If $u = x + 1$ then $\mathrm{d}u = \mathrm{d}x$

$$\sqrt{x^3 + x^2} = x\sqrt{x+1} = (u-1)\sqrt{u} = u^{1.5} - u^{0.5}$$

$$\begin{aligned} \int \sqrt{x^3 - x^2} \ \mathrm{d}x &= \int u^{1.5} - u^{0.5} \ \mathrm{d}u \\ &= \frac{2}{5}u^{2.5} - \frac{2}{3}u^{1.5} + c \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c \end{aligned}$$

12 If $x = \sin u$ then $\mathrm{d}x = \cos u \, \mathrm{d}u$

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{\cos u}$$

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}} \ \mathrm{d}x &= \int \frac{1}{\cos u} \cos u \, \mathrm{d}u \\ &= \int 1 \ \mathrm{d}u \\ &= u + c \\ &= \arcsin x + c \end{aligned}$$

Mixed Practice

1

$$\int_0^a 3x^2 - 4 \, dx = \left[\frac{3}{3}x^3 - 4x\right]_0^a$$
$$= a^3 - 4a$$

2

$$y = \int \frac{1}{4}x^{-0.5} \, dx$$
$$= \frac{1}{2}\sqrt{x} + c$$
$$y(4) = 3 = \frac{1}{2}(2) + c \text{ so } c = 2$$
$$y = \frac{1}{2}\sqrt{x} + 2$$

3

$$A = \int_0^{\frac{\pi}{6}} \cos x - \sin x \, dx$$
$$= 0.366 \text{ (GDC)}$$

4

$$\int \frac{3}{4}x^{-1} - \frac{1}{2}x^{-2} \, dx = \frac{3}{4}\ln x + \frac{1}{2x} + c$$

5

$$V = \pi \int_1^{2e} y^2 \, dx$$
$$= \pi \int_1^{2e} (\ln x)^2 \, dx$$
$$= 19.0 \text{ (GDC)}$$

6

$$V = \pi \int_0^{\frac{\pi}{2}} y^2 \, dx$$
$$= \pi \int_0^{\frac{\pi}{2}} (\cos x)^2 \, dx$$
$$= 2.47 \text{ (GDC)}$$

7 $x = y^2$

$$A = \int_1^2 x \, dy$$
$$= \int_1^2 y^2 \, dy$$
$$= \frac{7}{3}$$

8

$$f(x) = \int \frac{12}{2x-5}\,dx = \int \frac{6}{x-2.5}\,dx = 6\ln(x-2.5) + c$$
$$f(4) = 0 = 6\ln 1.5 + c$$
$$c = -6\ln 1.5$$
$$f(x) = 6\ln\left(\frac{x-2.5}{1.5}\right) = 6\ln\left(\frac{2x-5}{3}\right)$$

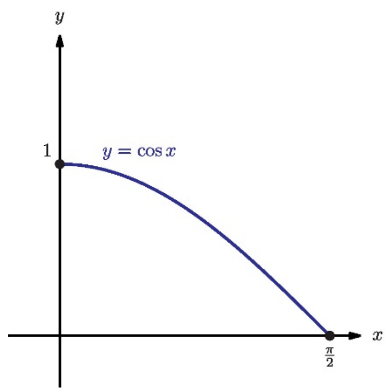
9

$$\int 3x^{0.5} - 2x^{-1.5}\,dx = 2x^{1.5} + 4x^{-0.5} + c$$

10

$$f'(x) = 3\cos x \sin^2 x$$
$$f(x) = \int 3\cos x \sin^2 x\ \,dx = \sin^3 x + c$$
$$f(\pi) = 2 = c$$
$$f(x) = \sin^3 x + 2$$

11 **a**



b $A(0,1),\ B\left(\frac{\pi}{2}, 0\right)$

c

The gradient is $\frac{-1}{\left(\frac{\pi}{2}\right)} = -\frac{2}{\pi}$

Line equation is $y = -\frac{2}{\pi}x + 1$

d

$$\text{Area} = \int_0^{\frac{\pi}{2}} \left(\cos x - 1 + \frac{2}{\pi}x\right)\,dx$$
$$= \left[\sin x - x + \frac{1}{\pi}x^2\right]_0^{\frac{\pi}{2}}$$
$$= 1 - \frac{\pi}{2} + \frac{\pi}{4}$$
$$= 1 - \frac{\pi}{4}$$

12 $f(x) = x^2 - kx$

Require that $\int_0^2 f(x)\ dx = 0$

$$\int_0^2 f(x)\ dx = \left[\frac{1}{3}x^3 - \frac{1}{2}kx^2\right]_0^2$$
$$= \frac{8}{3} - 2k = 0$$
$$k = \frac{4}{3}$$

13 a From GDC: $A\ (0,1), B\ (2,5), C\ (7,0)$

b Shaded region is the area of the triangle DBC where D has coordinates (2,0) plus the area between the curve and the x-axis for $0 \leqslant x \leqslant 2$.

$$\text{Shaded area} = \frac{1}{2}(5 \times 5) + \int_0^2 x^2 + 1\ dx$$
$$= \frac{103}{6}$$

14 a $a = \ln 11$

b $y = \ln(5x + 1)$ so $x = \frac{1}{5}(e^y - 1)$

$$\text{Area} = \int_0^{\ln 11} x\ dy$$
$$= \int_0^{\ln 11} \frac{1}{5}e^y - \frac{1}{5}\ dy$$
$$= 1.52\ \text{(GDC)}$$

c

$$\text{Volume} = \pi \int_0^{\ln 11} x^2\ dy$$
$$= \pi \int_0^{\ln 11} \frac{1}{25}(e^{2y} - 2e^y + 1)\ dy$$
$$= 5.33\ \text{(GDC)}$$

15 $y = \ln(x^2)$ so $y(1) = 0, y(e^2) = 4$

$$x^2 = e^y$$

$$\text{Volume} = \pi \int_0^4 x^2\ dy$$
$$= \pi \int_0^4 e^y\ dy$$
$$= \pi[e^y]_0^4$$
$$= \pi(e^4 - 1)$$

16 Curve intersection at $(1, 1)$

First curve: $y_1 = |x|$ so $x^2 = y_1^2$

Second curve: $y_2 = 2 - x^4$ so $x^2 = \sqrt{2 - y_2}$

$$\begin{aligned}\text{Volume} &= \pi \int_0^1 x^2 \, dy_1 + \pi \int_1^2 x^2 \, dy_2 \\ &= \pi \left(\int_0^1 y_1^2 \, dy_1 + \int_1^2 (2 - y_2)^{0.5} \, dy_2 \right) \\ &= \pi \left(\left[\frac{1}{3} y^3 \right]_0^1 + \left[-\frac{2}{3} (2 - y)^{1.5} \right]_1^2 \right) \\ &= \pi \left(\frac{1}{3} + \frac{2}{3} \right) \\ &= \pi\end{aligned}$$

17 Axis intercepts are $(\pm 3, 0)$ and $(0, \pm 2)$

Volume rotated about x-axis:

$$\begin{aligned}V_x &= \pi \int_{-2}^{2} y^2 \, dx \\ &= \pi \int_{-2}^{2} \left(4 - \frac{4}{9} x^2 \right) dx \\ &= \pi \left[4x - \frac{4}{27} x^3 \right]_{-2}^{2} \\ &= \pi \left(16 - \frac{64}{27} \right) \\ &= \frac{368}{27} \pi\end{aligned}$$

Volume rotated about y-axis:

$$\begin{aligned}V_y &= \pi \int_{-3}^{3} x^2 \, dy \\ &= \pi \int_{-3}^{3} \left(9 - \frac{9}{4} y^2 \right) dy \\ &= \pi \left[9y - \frac{1}{4} y^3 \right]_{-3}^{3} \\ &= \pi \left(54 - \frac{54}{4} \right) \\ &= \frac{81}{2} \pi\end{aligned}$$

18 **a**

$$\begin{aligned}\text{Volume} &= \pi \int_0^a y^2 \, dx \\ &= \pi \int_0^a \sec^2 x \, dx \\ &= \pi [\tan x]_0^a \\ &= \pi \tan a\end{aligned}$$

b $a = \frac{\pi}{4} \approx 0.786$

19

$$\begin{aligned}\text{Area} &= \int_0^4 y \ \mathrm{d}x \\ &= \frac{1}{2}\int_0^4 \frac{2x}{x^2+1}\ \mathrm{d}x \\ &= \left[\frac{1}{2}\ln(x^2+1)\right]_0^4 \\ &= \frac{1}{2}\ln 17\end{aligned}$$

20

$$\begin{aligned}\int_{\frac{3\pi}{2}}^{b}\cos x \ \mathrm{d}x &= [\sin x]_{\frac{3\pi}{2}}^{b} \\ &= \sin b + 1 \\ &= 1 - \frac{\sqrt{3}}{2} \\ \sin b &= -\frac{\sqrt{3}}{2} \text{ for some } \frac{3\pi}{2} < b < 2\pi \\ b &= \frac{5\pi}{3}\end{aligned}$$

21 a Curve to pass through $(0, 10), (20.5, 25), (50, 17.5), (55,18)$.

Simultaneous equations:

$$\begin{cases} d = 10 & (1) \\ (20.5)^3 a + (20.5)^2 b + 20.5c + d = 25 & (2) \\ 125000a + 2500b + 50c + d = 17.5 & (3) \\ 166375a + 3025b + 55c + d = 18 & (4)\end{cases}$$

Solving using GDC:

$a = 0.000545, b = -0.0582, c = 1.69, d = 10$

$y = 0.000545x^3 - 0.0582x^2 + 1.69x + 10$

b

$$\begin{aligned}V &= \pi\int_0^{55} y^2 \ \mathrm{d}x \\ &= 74\,400 \text{ cm}^3 \\ &= 74.4 \text{ litres}\end{aligned}$$

22

$$\int_5^8 2\mathrm{f}(x-3)\,\mathrm{d}x = 2\int_2^5 \mathrm{f}(x)\,\mathrm{d}x = 6$$

23 a $\mathrm{f}(x)$ is decreasing where $\mathrm{f}'(x) < 0$: $0 < x < d$

b $\mathrm{f}(x)$ is concave-up where $\mathrm{f}'(x)$ is increasing: $a < x < b$ and $x > c$

c

$$\int_a^0 \mathrm{f}'(x) \ \mathrm{d}x - \int_0^d \mathrm{f}'(x) \ \mathrm{d}x = \mathrm{f}(0) - \mathrm{f}(a) - \big(\mathrm{f}(d) - \mathrm{f}(0)\big)$$

$$= 2\mathrm{f}(0) - \mathrm{f}(a) - \mathrm{f}(d)$$
$$= 2\mathrm{f}(0) - 8 - 2 = 20$$
$$2\mathrm{f}(0) = 30$$
$$\mathrm{f}(0) = 15$$

24 a

Gradient is $\dfrac{h}{b-a}$

Line equation is $y = \dfrac{h}{b-a}(x-a)$

(or if $b = a$, the line is $x = a$)

Generally then,

$hx + (a-b)y = ah$

b Equivalently:

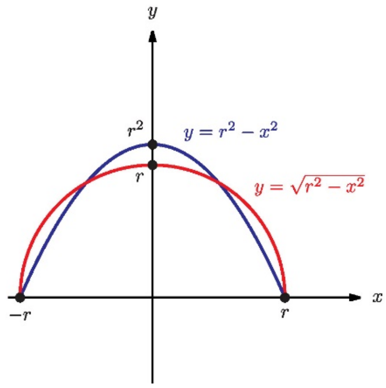
$x = \dfrac{b-a}{h}y + a$

Assuming $a \neq b$:

$$\text{Volume} = \pi\int_0^h x^2 \, \mathrm{d}y$$
$$= \pi\int_0^h \left(\frac{b-a}{h}y + a\right)^2 \mathrm{d}y$$
$$= \pi\left[\frac{h}{3(b-a)}\left(\frac{b-a}{h}y + a\right)^3\right]_0^h$$
$$= \frac{\pi h}{2(b-a)}(b^3 - a^3) \qquad (*)$$
$$= \frac{\pi h}{3}(b^2 + ab + a^2)$$

(Note that the case $b = a$, which would cause concern at step $(*)$ when the expression $(b-a)$ is cancelled in the result, would occur when the line is vertical and the resultant solid is a cylinder of radius a and height h. In that case, $b^2 + ab + a^2 = 3a^2$ and the volume given in the formula would correctly be $\pi a^2 h$.)

25 a



b Rearranging the curve formulae for the upper right quadrant of both curves:

Parabola: $x^2 = r^2 - y$

Circle: $x^2 = r^2 - y^2$

$$\text{Difference in volumes} = \pi\left(\int_0^r r^2 - y^2\ \mathrm{d}y - \int_0^{r^2} r^2 - y\ \mathrm{d}y\right)$$

$$= \pi\left(\left[r^2y - \frac{1}{3}y^3\right]_0^r - \left[r^2y - \frac{1}{2}y^2\right]_0^{r^2}\right)$$

$$= \pi\left(\frac{2}{3}r^3 - \frac{1}{2}r^4\right)$$

If the difference is zero:

$$\frac{2}{3}r^3 = \frac{1}{2}r^4$$

$$r = \frac{4}{3}$$

26 a $y = \ln x$ so $x = \mathrm{e}^y$

$$\text{Area} = \int_0^{\ln a} x\ \mathrm{d}y$$

$$= [\mathrm{e}^y]_0^{\ln a}$$

$$= a - 1$$

b The rectangle with vertices at the origin, $(a, 0)$, $(a, \ln a)$ and $(0, \ln a)$ has area $a \ln a$.

The part to the left of the curve $y = \ln x$ has area $a - 1$, by part **a**.

Then the part to the right of the curve, which is the area under the curve between $x = 1$ and $x = a$, must have the remainder of the area. That is,

$$\int_1^a \ln x\,\mathrm{d}x = a\ln a + 1 - a$$

12 Kinematics

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 12A

16 a $s(2) = 20 - 4 = 16 \text{ m}$

b

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = 10 - 2t$$
$$v(3) = 4 \text{ m s}^{-1}$$

c

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -2$$
$$a(4) = -2 \text{ m s}^{-2}$$

17

$$\begin{aligned} s(3) &= \int_0^3 v \, \mathrm{d}t \\ &= \int_0^3 \mathrm{e}^{-0.5t} \, \mathrm{d}t \\ &= [-2\mathrm{e}^{-0.5t}]_0^3 \\ &= 2 - 2\mathrm{e}^{-1.5} \approx 1.55 \text{ m} \end{aligned}$$

18 a

$$\begin{aligned} s(6) &= \int_0^6 v \, \mathrm{d}t \\ &= -48 \text{ m} \end{aligned}$$

b

$$\begin{aligned} d(6) &= \int_0^6 |v| \, \mathrm{d}t \\ &= 58.7 \text{ m} \end{aligned}$$

19 $a = 2t + 1$

$$v = \int a \, \mathrm{d}t = t^2 + t + c$$
$$v(0) = 3 = c$$
$$v = t^2 + t + 3$$
$$v(4) = 16 + 4 + 3 = 23 \text{ m s}^{-1}$$

20 **a** $v(0) = 256 \text{ m s}^{-1}$

b

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -4t^3$$
$$a(2) = -32 \text{ m s}^{-2}$$

c Bullet stops when $v = 0$

$$t^4 = 256$$
$$t = 4 \text{ s}$$

d Since all motion is in the same direction, distance travelled is the same as displacement.

$$\begin{aligned} s(4) &= \int_0^4 v \, \mathrm{d}t \\ &= \int_0^4 256 - t^4 \, \mathrm{d}t \\ &= \left[256t - \frac{1}{5}t^5\right]_0^4 \\ &= \frac{4096}{5} = 819.2 \text{ m} \end{aligned}$$

21

$$\begin{aligned} d(2) &= \int_0^2 |v| \, \mathrm{d}t \\ &= 3.16 \text{ (GDC)} \end{aligned}$$

22 **a** $t^2(6 - t) = 0$

Displacement is zero at $t = 0$ and $t = 6$ seconds.

b

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = 12t - 3t^2$$
$$3t(4 - t) = 0$$

Velocity is zero at $t = 0$ and $t = 4$ seconds.

c Displacement is a negative cubic, so has a local maximum at the second stationary point.

The maximum displacement for $x \geqslant 0$ therefore occurs at $t = 4$ seconds.

23 **a**

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \sin t + t \cos t$$

b $v(0) = 0 \text{ m s}^{-1}$

c

$$\begin{aligned} v\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2} + \frac{\pi}{4} \times \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2}(4 + \pi)}{8} \text{ m s}^{-1} \end{aligned}$$

23 d

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = 2\cos t - t\sin t$$
$$a(0) = 2 \text{ m s}^{-2}$$

e $v = 1$ at $t = 0.556, 1.57$ or 5.10 seconds (GDC)

f

$$d = \int_0^{2\pi} |v| \; \mathrm{d}t$$
$$= 13.3 \text{ m}$$

24 a

$$v = \int a \; \mathrm{d}t$$
$$= At + c$$
$$v(0) = c = u$$

So $v = At + u$

b

$$s = \int v \; \mathrm{d}t$$
$$= \frac{1}{2}At^2 + ut + k$$
$$s(0) = 0 = k$$
$$s = \frac{1}{2}At^2 + ut$$

25 $x = \sin \omega t$

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \omega \cos \omega t$$

$$\ddot{x} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = -\omega^2 \sin \omega t = -\omega^2 x$$

26 a

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -10 \text{ m s}^{-2}$$

b Maximum height occurs when $v = 0$, so at $t = 0.5$ seconds.

$$s = \int v \; \mathrm{d}t$$
$$= 5t - 5t^2 + s(0)$$
$$s = 5t(1 - t) + 60$$
$$s(0.5) = 61.25 \text{ m}$$

c $s = 0$ when $5t^2 - 5t - 60 = 0$

$$t^2 - t - 12 = 0$$
$$(t + 3)(t - 4) = 0$$

The ball hits the sea at $t = 4$ seconds.

26 d

$$d = \int_0^4 |v| \; \mathrm{d}t$$

Using the context, the ball rises 1.25 m and then falls 61.25 m,
so the total vertical distance travelled is 62.5 m.

e The model assumes no obstructions and no significant air resistance or air movement interfering with the projected movement of the ball.

The model also uses acceleration due to gravity of 10 m s^{-2}, which is an approximation.

27 a $v(0) = 18 \text{ m s}^{-1}$

b $18 - 2t^2 = 0$ for $t = 3$ seconds.

c

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -4t$$
$$a(2) = -8 \text{ m s}^{-2}$$

d

$$d(6) = \int_0^6 |v| \; \mathrm{d}t = 108 \text{ m}$$

e

$$s(6) = \int_0^6 v \; \mathrm{d}t = -36 \text{ m}$$

f Since the bicycle turns at $t = 3$, and travels 108 m in total, ending 36 m behind its start position, it must travel 36 m forwards for the first 3 seconds, then 72 m in the second 3 seconds.

It travels 36 m in the first 3 seconds.

28

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = -3t^2 + 12t - 2$$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -6t + 12$$

Maximum velocity occurs when $a = 0$: $t = 2$

$v(2) = -12 + 24 - 2 = 10 \text{ m s}^{-1}$

29

$$v(2) = \int_0^2 a\ \mathrm{d}t$$
$$= \int_0^2 2t\ \mathrm{d}t$$
$$= [t^2]_0^2$$
$$= 4\ \mathrm{m\ s^{-1}}$$

$$v(4) = v(2) + \int_2^4 a\ \mathrm{d}t$$
$$= 4 + \int_2^4 16t^{-2}\ \mathrm{d}t$$
$$= 4 + [-16t^{-1}]_2^4$$
$$= 4 + (8 - 4)$$
$$= 8\ \mathrm{m\ s^{-1}}$$

30 Root of the equation $v = \frac{1}{t} - 1$ occurs at $t = 1$.

$$d = \int_{0.5}^{2} |v|\ \mathrm{d}t$$
$$= \int_{0.5}^{1} v\ dt - \int_{1}^{2} v\ dt$$
$$= [\ln t - t]_{0.5}^{1} - [\ln t - t]_1^2$$
$$= -1 - \ln 0.5 + 0.5 - \ln 2 + 2 - 1$$
$$= 0.5 - \ln(0.5 \times 2)$$
$$= 0.5\ \mathrm{m}$$

31 $v = 50 - 40\mathrm{e}^{-0.02s}$

a $v|_{s=0} = 10$

The speed at the start of the road is $10\ \mathrm{m\ s^{-1}}$

b $v|_{s=100} = 50 - 40\mathrm{e}^{-2} = 44.6\ \mathrm{m\ s^{-1}}$

c

$$a = \frac{\mathrm{d}v}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s} = (50 - 40\mathrm{e}^{-0.02s})(0.8\mathrm{e}^{-0.02s})$$

$a|_{s=10} = 40\mathrm{e}^{-0.2} - 32\mathrm{e}^{-0.4} = 11.3\ \mathrm{m\ s^{-2}}$

32 **a** $s(10) = 0\ \mathrm{m}$

b

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = 10 - 2t$$

The movement is symmetrical, turning back at $t = 5$

$d(10) = 2s(5) = 2 \times 25 = 50\ \mathrm{m}$

33 From the GDC:

a Maximum speed $= 6.06\ \mathrm{m\ s^{-1}}$

b Minimum speed $= 0\ \mathrm{m\ s^{-1}}$

33 c

$$\text{Ave speed} = \frac{1}{4}\int_0^4 |v| \text{ d}t = 2.96 \text{ m s}^{-1}$$

34 a i

$$\begin{aligned} v &= \int a \text{ d}t \\ &= \frac{1}{2}At^2 + v(0) \\ &= \frac{1}{2}At^2 + u \end{aligned}$$

ii

$$\begin{aligned} s &= \int v \text{ d}t \\ &= \frac{1}{6}At^3 + ut + s(0) \\ &= \frac{1}{6}At^3 + ut \end{aligned}$$

b Since acceleration is positive from $t = 0$, $v > u > 0$ for all $t > 0$.

Then average speed is total distance divided by time taken.

$$\begin{aligned} \text{Average speed} &= \frac{s}{t} \\ &= \frac{1}{6}At^2 + u \\ &= \frac{1}{3}\left(\frac{1}{2}At^2 + u\right) + \frac{2}{3}u \\ &= \frac{1}{3}v + \frac{2}{3}u \\ &= \frac{v + 2u}{3} \end{aligned}$$

35 Let Jane's distance from Aisla's start line be s_J and Aisla's distance from her start line be s_A

$$s_J = \int v_J \text{ d}t = \frac{1}{2}t^2 + 2t + s_J(0) = \frac{1}{2}t^2 + 2t + 42$$

$$s_A = \int v_A \text{ d}t = \frac{1}{3}t^3 + s_A(0) = \frac{1}{3}t^3$$

Aisla passes Jane when $S_A = s_J$

From GDC, the solution to $\frac{1}{2}t^2 + 2t + 42 = \frac{1}{3}t^3$ is $t = 6$ seconds.

36 Let the position of the first ball be given by

$$x_1(t) = \int v \; \mathrm{d}t = 5t - 5t^2$$

Let the position of the second ball from $t = 1$ be given by x_2

$$x_2(t) = x_1(t - 0.5) = 5(t - 0.5) - 5(t - 0.5)^2$$

The two collide when $x_1 = x_2$

$$5t(1 - t) = 5(t - 0.5)(1.5 - t)$$

From GDC, or from considering the symmetry, since the first ball is back at zero displacement at $t = 1$, the intersection occurs when $t = 0.75$.

$$x_1(0.75) = x_2(0.75) = 0.9375 \text{ m}$$

37

$$a = v\frac{\mathrm{d}v}{\mathrm{d}s} = s^3 + s$$

Separating variables:

$$\int v \; \mathrm{d}v = \int s^3 + s \; \mathrm{d}s$$

$$\frac{1}{2}v^2 = \frac{1}{4}s^4 + \frac{1}{2}s^2 + c$$

$$v = \sqrt{\frac{1}{2}s^4 + s^2 + k}$$

38 Consider an object which moves a distance a in time b, and then a distance c in time d, with the second part of the movement being faster.

Then the average speed in the first period of movement is $\frac{\text{distance}}{\text{time}} = \frac{a}{b}$

The (greater) average speed in the second period of movement is $\frac{\text{distance}}{\text{time}} = \frac{c}{d}$

Then the average speed over the whole period must be between these two values.

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{a + c}{b + d}$$

Therefore,

$$\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$$

Exercise 12B

9 **a** $\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \begin{pmatrix} 1.5\cos 3t \\ 4 \end{pmatrix}$

b $|\mathbf{v}(4)| = \left|\begin{pmatrix} 1.5\cos(12) \\ 4 \end{pmatrix}\right| = 4.20 \text{ m s}^{-1}$

10 a $\mathbf{v}(0) = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

Angle with **i** is $\tan^{-1}\left(\frac{7}{4}\right) \approx 60.3°$

b

$$\mathbf{a} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$
$$|\mathbf{a}| = \sqrt{3^2 + (-2)^2} \approx 3.61 \text{ m s}^{-2}$$

11 a $|\mathbf{v}(3)| = \mathrm{e}^{-3}\sqrt{3^2 + 4^2} = 5\mathrm{e}^{-3} \approx 0.249 \text{ m s}^{-1}$

b $\mathbf{r}(t) = \int_0^t \mathbf{v}(\tau)\ \mathrm{d}\tau = \begin{pmatrix} -3\mathrm{e}^{-t} \\ -4\mathrm{e}^{-t} \end{pmatrix} - \mathbf{s}(0) = \begin{pmatrix} -3\mathrm{e}^{-t} \\ -4\mathrm{e}^{-t} \end{pmatrix}$

c $|\mathbf{s}(3)| = 5\mathrm{e}^{-3} \approx 0.249 \text{ m}$

12 $\mathbf{v}(2) = \begin{pmatrix} 7 \\ 9.6 \end{pmatrix}$

a

Angle is $\tan^{-1}\left(\frac{9.6}{7}\right) = 53.9°$

b $|\mathbf{v}(2)| = \sqrt{7^2 + 9.6^2} = 11.9 \text{ m s}^{-1}$

c

$$\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \begin{pmatrix} 3 \\ -0.2t \end{pmatrix}$$
$$|\mathbf{a}(2)| = \sqrt{3^2 + (-0.6)^2} = 3.06 \text{ m s}^{-2}$$

13 a

$$\mathbf{v}(5) = \begin{pmatrix} -5 \\ 16 \end{pmatrix}$$
$$|\mathbf{v}(5)| = \sqrt{(-5)^2 + 16^2} = 16.8 \text{ m s}^{-1}$$

b

$\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, which is independent of t

$$a = \sqrt{(-2)^2 + 3^2} = 3.61 \text{ m s}^{-2}$$

14 a

$$\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

The acceleration is 9.8 m s^{-2} directed downwards.

b

$$\mathbf{v}(0) = \begin{pmatrix} 16 \\ 12 \end{pmatrix}$$
$$|\mathbf{v}(0)| = \sqrt{16^2 + 12^2} = 20 \text{ m s}^{-1}$$

c Speed is given by $|\mathbf{v}(t)| = \sqrt{16^2 + (12 - 9.8t)^2}$

The minimum speed is therefore 16 m s^{-1}

15 a

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{d}t = \begin{pmatrix} 9t + c_x \\ 7t - 4.9t^2 + c_y \end{pmatrix}$$

$$\mathbf{r}(0) = \mathbf{0} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} 9t \\ 7t - 4.9t^2 \end{pmatrix}$$

b Assuming the particle lands when the y component of its displacement returns to zero:

$$7t - 4.9t^2 = 0$$

$$0.7t(10 - 7t) = 0$$

$$t = \frac{10}{7} \approx 1.43 \text{ s}$$

$$\mathbf{r}\left(\frac{10}{7}\right) = \begin{pmatrix} \frac{90}{7} \\ 0 \end{pmatrix}$$

The particle lands approximately 12.9 m from the origin.

c

$$\mathbf{v}\left(\frac{10}{7}\right) = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$$

$$|\mathbf{v}(0)| = \left|\begin{pmatrix} 9 \\ 7 \end{pmatrix}\right| = \left|\mathbf{v}\left(\frac{10}{7}\right)\right|$$

The particle's speed at the origin is the same as its speed when it lands.

16 a i $\mathbf{v}(0) = \begin{pmatrix} 20 \\ 10 \end{pmatrix}$

The velocity has angle $\tan^{-1}\left(\frac{10}{20}\right) = 26.6°$ above the horizontal

ii $\mathbf{v}(2) = \begin{pmatrix} 20 \\ -9.6 \end{pmatrix}$

The velocity has angle $\tan^{-1}\left(-\frac{9.6}{20}\right) = -25.6°$
that is, $25.6°$ below the horizontal,

b Maximum height will occur when the vertical component of velocity is zero.

$$10 - 9.8t = 0$$

$$t = \frac{10}{9.8} \approx 1.02 \text{ s}$$

c

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 20t + c_x \\ 10t - 4.9t^2 + c_y \end{pmatrix}$$

$$\mathbf{r}(0) = \mathbf{0} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} 20t \\ 10t - 4.9t^2 \end{pmatrix}$$

$$\mathbf{r}\left(\frac{10}{9.8}\right) = \begin{pmatrix} \frac{200}{9.8} \\ \frac{100}{9.8} - \frac{50}{9.8} \end{pmatrix}$$

The maximum height is $\dfrac{50}{9.8} \approx 5.10$ m

17 a

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 12t + c_x \\ 9t - 4.9t^2 + c_y \end{pmatrix}$$

$$\mathbf{r}(0) = \mathbf{0} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} 12t \\ 9t - 4.9t^2 \end{pmatrix}$$

b The height of the projectile is $9t - 4.9t^2 = 0.1t(90 - 49t)$

The projectile has zero height at $t = 0$ and $t = \frac{90}{49} \approx 1.84$ seconds.

$$\mathbf{r}\left(\frac{90}{49}\right) = \begin{pmatrix} \frac{1080}{49} \\ 0 \end{pmatrix}$$

The projectile lands $\frac{1080}{49} \approx 22.0$ m from the origin

18 a $|\mathbf{r}(t)| = \sqrt{25\cos^2 3t + 25\sin^2 3t} = 5\sqrt{\cos^2 3t + \sin^2 3t} = 5$, independent of t

Since the distance from the origin is 5 m at all times, the object must only move on the circle of that radius around the origin.

b

$$\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \begin{pmatrix} -15\sin 3t \\ 15\cos 3t \end{pmatrix}$$

$$\mathbf{v}(t) \cdot \mathbf{r}(t) = -75\sin 3t\cos 3t + 75\sin 3t\cos 3t = 0$$

So velocity and position vectors are at every instant perpendicular.

c $|\mathbf{v}(t)| = \sqrt{(-15\sin 3t)^2 + (15\cos 3t)^2} = 15\text{ m s}^{-1}$, independent of t

The object has a constant speed 15 m s^{-1}

19 a $\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \begin{pmatrix} -6\sin(0.5t) \\ 6\cos(0.5t) \end{pmatrix}$

b $|\mathbf{v}(t)| = \sqrt{(-6\sin(0.5t))^2 + (6\cos(0.5t))^2} = 6\text{ m s}^{-1}$, independent of t

c $\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \begin{pmatrix} -3\cos(0.5t) \\ -3\sin(0.5t) \end{pmatrix} = -\frac{1}{4}\mathbf{r}(t)$

d $|\mathbf{a}(t)| = \sqrt{(-3\cos(0.5t))^2 + (-3\sin(0.5t))^2} = 3\text{ m s}^{-2}$, independent of t

20 a

$|\mathbf{v}(t)| = \sqrt{(0.9\cos(3t))^2 + (0.9\sin(3t))^2} = 0.9\text{ m s}^{-1}$, independent of t

b

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 0.3\sin 3t + c_x \\ -0.3\cos 3t + c_y \end{pmatrix}$$

$$\mathbf{r}(0) = \begin{pmatrix} 0 \\ -0.3 \end{pmatrix} = \begin{pmatrix} c_x \\ -0.3 + c_y \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} 0.3\sin 3t \\ -0.3\cos 3t \end{pmatrix}$$

Then $|\mathbf{r}(t)| = 0.3$ m, independent of t, and so the object moves on a circle of radius 0.3 m about the origin.

20 c

$$\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \begin{pmatrix} -2.7\sin 3t \\ 2.7\cos 3t \end{pmatrix} = -9\mathbf{r}(t)$$
$$k = -9$$

d The acceleration vector is antiparallel to the position vector; that is, it always points towards the origin from the position of the object.

21 a $|\mathbf{r}(t)| = \sqrt{0.25\cos^2 4t + 0.25\sin^2 4t} = 0.5\sqrt{\cos^2 4t + \sin^2 4t} = 0.5$, independent of t

Since the distance from the origin is 0.5 m at all times, the object must only move on the circle of that radius around the origin.

b $\mathbf{r}(0) = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$

c The period of the oscillation is $T = \frac{2\pi}{4} \approx 1.57$ seconds.

d

$$\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \begin{pmatrix} -2\sin(4t) \\ 2\cos(4t) \end{pmatrix}$$
$$|\mathbf{v}(t)| = 2$$
$$\frac{2\pi r}{T} = \frac{2\pi(0.5)}{2\pi/4} = 2 = v(t)$$

22 a

$$\mathbf{r_1}(t) = \int \mathbf{v_1}(t)\ \mathrm{d}t = \begin{pmatrix} 4t - 0.75t^2 + c_x \\ 0.1t^3 + 5t + c_y \end{pmatrix}$$
$$\mathbf{r_1}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$
$$\mathbf{r_1}(t) = \begin{pmatrix} 4t - 0.75t^2 \\ 0.1t^3 + 5t \end{pmatrix}$$

b Let the second object have position given by $\mathbf{r_2}(t)$ for $t \geqslant 2$

$$\mathbf{r_2}(t) = \mathbf{r_1}(t-2)$$
$$\mathbf{r_2}(5) - \mathbf{r_1}(5) = \mathbf{r_1}(3) - \mathbf{r_1}(5)$$
$$= \begin{pmatrix} 5.25 \\ 17.7 \end{pmatrix} - \begin{pmatrix} 1.25 \\ 37.5 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ -19.8 \end{pmatrix}$$

The distance between the objects is $\sqrt{4^2 + (-19.8)^2} = 20.2$ m

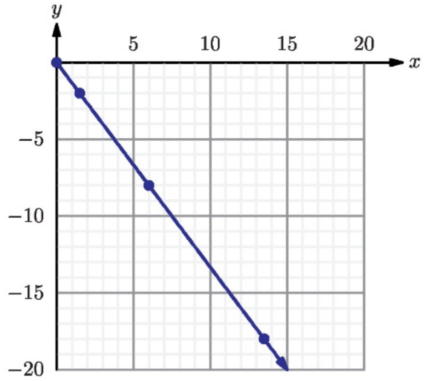
23 a

$$\mathbf{v}(t) = \int \mathbf{a}(t)\ \mathrm{d}t = \begin{pmatrix} 3t + c_x \\ -4t + c_y \end{pmatrix}$$
$$\mathbf{v}(0) = \mathbf{0} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$
$$\mathbf{v}(t) = \begin{pmatrix} 3t \\ -4t \end{pmatrix}$$
$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 1.5t^2 + k_x \\ -2t^2 + k_y \end{pmatrix}$$
$$\mathbf{r}(0) = \mathbf{0} = \begin{pmatrix} k_x \\ k_y \end{pmatrix}$$
$$\mathbf{r}(t) = \begin{pmatrix} 1.5t^2 \\ -2t^2 \end{pmatrix}$$

23 b $v(t) = \sqrt{(3t)^3 + (-4t)^2} = 5t$

Speed $v(4) = 20$ so it takes 4 seconds to reach speed 20 m s^{-1}.

c $\mathbf{r} = \frac{t^2}{2}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$



The object follows a straight-line trajectory $4x + 3y = 0$;
it travels faster the further it moves from the origin.

24 a

$$\mathbf{v}(t) = \int \mathbf{a}(t)\ \mathrm{d}t = \begin{pmatrix} -2t + c_x \\ t + c_y \end{pmatrix}$$

$$\mathbf{v}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

$$\mathbf{v}(t) = \begin{pmatrix} 1 - 2t \\ 3 + t \end{pmatrix}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} t - t^2 + k_x \\ 3t + 0.5t^2 + k_y \end{pmatrix}$$

$$\mathbf{r}(0) = \mathbf{0} = \begin{pmatrix} k_x \\ k_y \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} t - t^2 \\ 3t + 0.5t^2 \end{pmatrix}$$

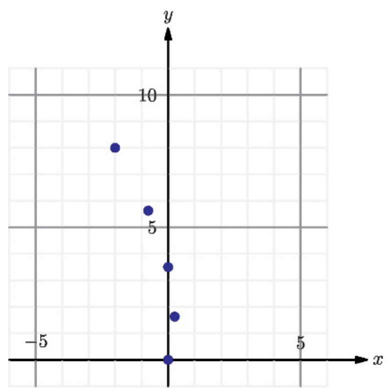
b $\mathbf{v}(2) = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

The angle of $\mathbf{v}(0)$ is $\tan^{-1}\left(\frac{3}{1}\right) = 71.6°$

The angle of $\mathbf{v}(2)$ is $180° + \tan^{-1}\left(\frac{5}{-3}\right) = 121°$

The angle between the vectors is 49.4°

c The trajectory does not follow a straight line.



25 Assuming gravity is the only significant force acting on the projectile, the object will experience only an accelerative element downwards, which can be designated as g_X.

$$\mathbf{a} = \begin{pmatrix} 0 \\ -g_X \end{pmatrix}$$

$$\mathbf{v}(t) = \int \mathbf{a}(t)\ \mathrm{d}t = \begin{pmatrix} c_x \\ -g_X t + c_y \end{pmatrix}$$

$$\mathbf{v}(0) = \begin{pmatrix} 12 \\ 9 \end{pmatrix} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

$$\mathbf{v}(t) = \begin{pmatrix} 12 \\ 9 - g_X t \end{pmatrix}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 12t + k_x \\ 9t - 0.5 g_X t^2 + k_y \end{pmatrix}$$

$$\mathbf{r}(0) = \mathbf{0} = \begin{pmatrix} k_x \\ k_y \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} 12t \\ 9t - 0.5 g_X t^2 \end{pmatrix}$$

The maximum value for $9t - 0.5g_X t^2$ is 8.6, which must occur when the vertical element of velocity is zero: $g_X t = 9$

$$t = \frac{9}{g_X}$$

Substituting:

$$\frac{81}{g_X} - \frac{40.5}{g_X} = 8.6$$

$$g_X = \frac{40.5}{8.6} \approx 4.7 \text{ m s}^{-2}$$

26 Let the position at $t = 0$ be given by $\mathbf{r} = \mathbf{0}$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 1.5t^2 - 5t + k_x \\ 4t - 0.5t^2 + k_y \end{pmatrix}$$

$$\mathbf{r}(0) = \mathbf{0} = \begin{pmatrix} k_x \\ k_y \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} 1.5t^2 - 5t \\ 4t - 0.5t^2 \end{pmatrix}$$

Solving for $\mathbf{r} = \mathbf{0}$:

$$\begin{pmatrix} 0.5t(3 - 10t) \\ 0.5t(8 - t) \end{pmatrix}$$

The horizontal component equals 0 at $t = 0$ or 0.3

The vertical component equals 0 at $t = 0$ or 8

The only occasion that both equal 0 at the same time is at the start of the motion, so the object never returns to its start position.

27 Assuming the particles land when the vertical component of their positions matches the initial value of zero.

$$\mathbf{r_1}(t) = \int \mathbf{v_1}(t)\ \mathrm{d}t = \begin{pmatrix} 26t + c_x \\ 21t - 2.45t^2 + c_y \end{pmatrix}$$

$$\mathbf{r_1}(0) = \mathbf{0} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

$$\mathbf{r_1}(t) = \begin{pmatrix} 26t \\ 21t - 2.45t^2 \end{pmatrix}$$

$$\mathbf{r_2}(t) = \int \mathbf{v_2}(t)\ \mathrm{d}t = \begin{pmatrix} 30t + k_x \\ bt - 2.45t^2 + k_y \end{pmatrix}$$

$$\mathbf{r_2}(0) = \mathbf{0} = \begin{pmatrix} k_x \\ k_y \end{pmatrix}$$

$$\mathbf{r_2}(t) = \begin{pmatrix} 30t \\ bt - 2.45t^2 \end{pmatrix}$$

The first particle lands when $21t - 2.45t^2 = 0$ so $t = \dfrac{21}{2.45}$

The second particle lands when $bt - 2.45t^2 = 0$ so $t = \dfrac{b}{2.45}$

Setting the horizontal component of each particle position at the time of landing to be equal:

$$26 \times \frac{21}{2.45} = 30 \times \frac{b}{2.45}$$

$$b = \frac{26 \times 21}{30} = 18.2$$

28 a $\mathbf{a} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$

$$\mathbf{v_1}(t) = \int \mathbf{a}(t)\ \mathrm{d}t = \begin{pmatrix} c_x \\ -9.8t + c_y \end{pmatrix}$$

$$\mathbf{v_1}(0) = \begin{pmatrix} 12 \\ 30 \end{pmatrix} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

$$\mathbf{v_1}(t) = \begin{pmatrix} 12 \\ 30 - 9.8t \end{pmatrix}$$

$$\mathbf{r_1}(t) = \int \mathbf{v_1}(t)\ \mathrm{d}t = \begin{pmatrix} 12t + k_x \\ 30t - 4.9t^2 + k_y \end{pmatrix}$$

$$\mathbf{r_1}(0) = \mathbf{0} = \begin{pmatrix} k_x \\ k_y \end{pmatrix}$$

$$\mathbf{r_1}(t) = \begin{pmatrix} 12t \\ 30t - 4.9t^2 \end{pmatrix}$$

b

$$\mathbf{v_2}(t) = \int \mathbf{a}(t)\ \mathrm{d}t = \begin{pmatrix} c_x \\ -9.8(t-1) + c_y \end{pmatrix}$$

$$\mathbf{v_2}(1) = \begin{pmatrix} 20 \\ b \end{pmatrix} = \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

$$\mathbf{v_2}(t) = \begin{pmatrix} 20 \\ b - 9.8(t-1) \end{pmatrix}$$

$$\mathbf{r_2}(t) = \int \mathbf{v_2}(t)\ \mathrm{d}t = \begin{pmatrix} 20(t-1) + k_x \\ b(t-1) - 4.9(t-1)^2 + k_y \end{pmatrix}$$

$$\mathbf{r_2}(1) = \mathbf{0} = \begin{pmatrix} k_x \\ k_y \end{pmatrix}$$

$$\mathbf{r_2}(t) = \begin{pmatrix} 20(t-1) \\ b(t-1) - 4.9(t-1)^2 \end{pmatrix}$$

28 c At some moment $t > 1$, $\mathbf{r_1}(t) = \mathbf{r_2}(t)$

$$\begin{pmatrix} 12t \\ 30t - 4.9t^2 \end{pmatrix} = \begin{pmatrix} 20(t-1) \\ b(t-1) - 4.9(t-1)^2 \end{pmatrix}$$

$$\begin{cases} 12t = 20t - 20 & (1) \\ 30t - 4.9t^2 = b(t-1) - 4.9(t-1)^2 & (2) \end{cases}$$

From (1): $t = 2.5$

Substituting into (2):

$$1.5b = 75 - 4.9(2.5^2 - 1.5^2) = 55.4$$
$$b = 36.9$$

d

$$\mathbf{r_1}(2.5) = \begin{pmatrix} 30 \\ 44.4 \end{pmatrix}$$
$$|\mathbf{r_1}(2.5)| = \sqrt{30^2 + 44.4^2} = 53.6 \text{ m}$$

29 a Let the particles have position vectors $\mathbf{r_1}$ and $\mathbf{r_2}$. For $t \geqslant 10$:

$$\mathbf{r_1}(t) = \begin{pmatrix} 120t \\ 90t - 4.9t^2 \end{pmatrix}$$
$$\mathbf{r_2}(t) = \mathbf{r_1}(t-10) = \begin{pmatrix} 120t - 1200 \\ 90t - 900 - 4.9t^2 + 98t - 490 \end{pmatrix}$$

Then the distance between then for $t \geqslant 10$ is

$$|\mathbf{r_1}(t) - \mathbf{r_1}(t-10)| = \left|\begin{pmatrix} 1200 \\ 1390 - 98t \end{pmatrix}\right|$$
$$= \sqrt{1200^2 + (1390 - 98t)^2}$$

b The first projectile lands when $(\mathbf{r_1})_y = 0$: $t(90 - 4.9t) = 0$ so $t \approx 18.4$ seconds.

Let $D = |\mathbf{r_1}(t) - \mathbf{r_1}(t-10)|^2$, the square of the distance between the projectiles.

$D = 1200^2 + (1390 - 98t)^2$

Interpreting this completed square form, D^2 (and therefore also D) is a minimum when

$t = \frac{1390}{98} \approx 14.2$.

30 a $|\mathbf{r}(t)| = \sqrt{r^2 \cos^2 \omega t + r^2 \sin^2 \omega t} = r$

The distance from the particle's position to the origin is a constant r,
so it moves only on the circle of radius r about the origin.

b $\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \begin{pmatrix} -r\omega \sin \omega t \\ r\omega \cos \omega t \end{pmatrix}$

$$v(t) = |\mathbf{v}(t)| = \sqrt{r^2\omega^2 \sin^2 \omega t + r^2\omega^2 \cos^2 \omega t} = r\omega$$

c $\mathbf{r}(0) = \begin{pmatrix} r \\ 0 \end{pmatrix}$

The particle returns to its initial position when $\begin{pmatrix} r\cos \omega t \\ r \sin \omega t \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$

This occurs for all $\omega t = 2n\pi$ for integer n.

It first does so when $\omega t = 2\pi$

so $t = \dfrac{2\pi}{\omega}$

30 d $\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \begin{pmatrix} -r\omega^2 \cos \omega t \\ -r\omega^2 \sin \omega t \end{pmatrix}$

$$a(t) = |\mathbf{a}(t)| = \sqrt{r^2\omega^4 \sin^2 \omega t + r^2\omega^4 \cos^2 \omega t} = r\omega^2 = \frac{r^2\omega^2}{r} = \frac{(v(t))^2}{r}$$

31 a

$$\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \begin{pmatrix} -r\omega \sin \omega t \\ r\omega \cos \omega t \end{pmatrix}$$
$$\mathbf{v}(t) \cdot \mathbf{r}(t) = -r^2\omega \sin \omega t \cos \omega t + r^2\omega \sin \omega t \cos \omega t = 0$$

so the velocity vector is perpendicular to the displacement vector.

b $\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \begin{pmatrix} -r\omega^2 \cos \omega t \\ -r\omega^2 \sin \omega t \end{pmatrix} = -\omega^2\mathbf{r}(t)$

The acceleration is directed parallel to, and in the opposite direction to, the displacement; that is, the acceleration is directed towards the centre of the circle.

Mixed Practice

1

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = 3 - 0.18t^2$$
$$v(8.6) = -10.3 \text{ m s}^{-1}$$
$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -0.36t$$
$$a(8.6) = -3.10 \text{ m s}^{-2}$$

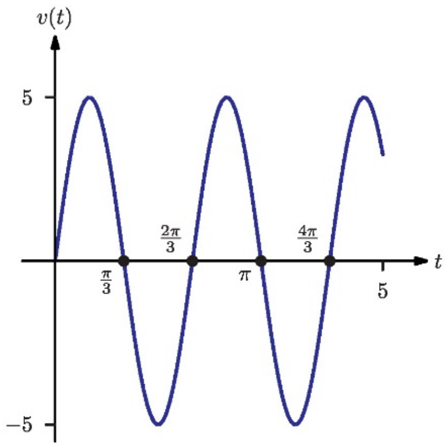
2 a

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -8\mathrm{e}^{-2t}$$
$$a(2) = -0.147 \text{ m s}^{-2}$$

b

$$s(2) = \int_0^2 v \ \mathrm{d}t$$
$$= [-2\mathrm{e}^{-2t}]_0^2$$
$$= 2 - 2\mathrm{e}^{-4} \approx 1.96 \text{ m}$$

3 a



3 **b** Direction change will happen when $v(t) = 0$

$$t = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \approx 1.05, 2.09, 3.14, 4.19 \text{ seconds}$$

c

$$d(5) = \int_0^5 |v(t)| \, \mathrm{d}t$$
$$= 16.3 \text{ m}$$

4 **a** $\mathbf{r}(0) = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$

b Circle radius is 7 cm

c $\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \begin{pmatrix} -21 \sin 3t \\ 21 \cos 3t \end{pmatrix}$

speed $= |\mathbf{v}(t)| = 21 \text{ cm s}^{-1}$

5 **a** $\mathbf{v}(0) = \begin{pmatrix} 16 \\ 14 \end{pmatrix}$ so initial speed is $\sqrt{16^2 + 14^2} \approx 21.3 \text{ m s}^{-1}$

b $\mathbf{v}(2) = \begin{pmatrix} 16 \\ -5.6 \end{pmatrix}$; the vertical component is negative so the projectile is on the way down.

c Taking initial position as the origin,

displacement $\mathbf{r}(t) = \int_0^t \mathbf{v}(\tau) \, \mathrm{d}\tau = \begin{pmatrix} 16t \\ 14t - 4.9t^2 \end{pmatrix}$

d $|\mathbf{r}(2)| = \sqrt{32^2 + (28 - 19.6)^2} \approx 33.1 \text{ m}$

6 **a** $\mathbf{v}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ so initial speed is 2 cm s^{-1}

b

$$\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \begin{pmatrix} 3 \\ -\mathrm{e}^{-0.5t} \end{pmatrix}$$
$$\mathbf{a}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

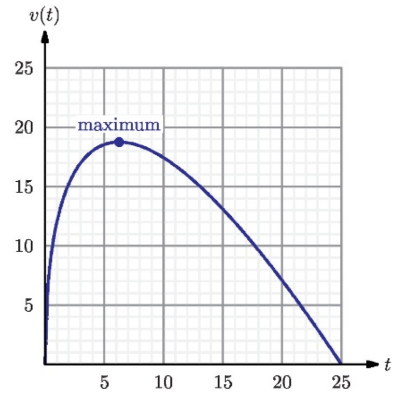
7 **a**

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = 12 - 6t^2$$
$$a(2.7) = -31.7 \text{ cm s}^{-1}$$

b

$$s(1.3) = \int_0^{1.3} v \, \mathrm{d}t$$
$$= \left[6t^2 - \frac{1}{2}t^4 - t\right]_0^{1.3}$$
$$\approx 7.41 \text{ cm}$$

8 **a**



b **i**

$$d = \int_0^9 15\sqrt{t} - 3t \, \mathrm{d}t$$

ii (From GDC) $d = 149$ m

9 **a**

$$\ddot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\dot{x} = 9 - 6t$$
$$\ddot{x}(3) = -9 \text{ m s}^{-2}$$

b

$$\begin{aligned} x &= \int_0^t \dot{x}(u)\ \mathrm{d}u + x(0) \\ &= \left[\frac{9}{2}u^2 - u^3\right]_0^t + x(0) \\ &= \frac{9}{2}t^2 - t^3 + 5 \\ x(3) &= 18.5 \text{ m} \end{aligned}$$

10 **a**

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 1.5t^2 + k_x \\ -4\mathrm{e}^{-0.5t} + k_y \end{pmatrix}$$
$$\boldsymbol{r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k_x \\ k_y - 4 \end{pmatrix} \text{ so } \boldsymbol{r}(t) = \begin{pmatrix} 1.5t^2 \\ 4 - 4\mathrm{e}^{-0.5t} \end{pmatrix}$$

b

$$\mathbf{r}(2) = \begin{pmatrix} 6 \\ 4 - 4\mathrm{e}^{-1} \end{pmatrix}$$
$$s(2) = \sqrt{36 + (4 - 4e^{-1})^2} \approx 6.51 \text{ cm}$$

11 **a**

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} -3\cos t + k_x \\ 4\sin t + k_y \end{pmatrix}$$
$$\mathbf{r}(0) = \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} k_x - 3 \\ k_y \end{pmatrix}$$
$$\text{so } \mathbf{r}(t) = \begin{pmatrix} -3\cos t \\ 4\sin t \end{pmatrix}$$

Worked solutions

11 b $\mathbf{r}(2\pi) = \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \mathbf{r}(0)$

The object returns to its start point at $t = 2\pi$

c $|\mathbf{r}(t)|^2 = 9\cos^2 t + 16\sin^2 t = 9 + 7\sin^2 t$

This clearly has a maximum at $9 + 7 = 16$ so the maximum value of $|\mathbf{r}(t)|$ is 4 m.

d $\mathbf{v}\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1.5\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}, \mathbf{r}\left(\frac{\pi}{4}\right) = \begin{pmatrix} -1.5\sqrt{2} \\ 2\sqrt{2} \end{pmatrix}$

Then the angle between velocity and displacement is θ where

$$\cos\theta = \frac{\mathbf{v}\left(\frac{\pi}{4}\right)\cdot\mathbf{r}\left(\frac{\pi}{4}\right)}{\left|\mathbf{v}\left(\frac{\pi}{4}\right)\right|\left|\mathbf{r}\left(\frac{\pi}{4}\right)\right|} = \frac{3.5}{\sqrt{12.5}\sqrt{12.5}} = \frac{7}{25}$$

$$\theta = \cos^{-1}(0.28) \approx 73.7°$$

12 a $\mathbf{v}(t) = \begin{pmatrix} 2 \\ 17 - 9.8t \end{pmatrix}$ m s^{-1}

b $\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 2t \\ 17t - 4.9t^2 \end{pmatrix} + \mathbf{r}_0$

Assume that the initial position is at the origin so $\mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\mathbf{r}(t) = \begin{pmatrix} 2t \\ 17t - 4.9t^2 \end{pmatrix}$$

The y-component will return to zero when $17t - 4.9t^2 = 0$

$$t = 0 \text{ or } \frac{17}{4.9} \approx 3.47 \text{ seconds}$$

c $|\mathbf{r}(t)| = \sqrt{(2t)^2 + (17t - 4.9t^2)^2}$

d From GDC, maximum $|\mathbf{r}(t)|$ is $\mathbf{r}(1.78) \approx 15.2$ m

13 a $\mathbf{r}(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

b $\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \begin{pmatrix} -8\sin 2t \\ 20\cos 4t \end{pmatrix}$

c $|\mathbf{v}(t)|^2 = 64\sin^2 2t + 400\cos^2 4t$

From GDC, this has maximum at approximately 21.5 m s^{-1}

d The particle will pass through the origin at t such that:

$$\cos 2t = 0 = \sin 4t$$
$$\sin 4t = 2\sin 2t\cos 2t$$

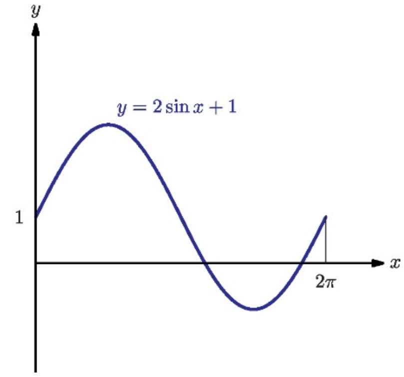
Therefore any t for which $\cos 2t = 0$ will find the particle at the origin.

The particle will be at the origin at $t = \frac{(2n+1)\pi}{4}$ for any integer n.

e The least t for which both $\cos 2t$ and $\sin 4t$ have completed full periods is $t = \pi$

The particle first returns to its starting position at $t = 3.14$ seconds.

14 a



b

$$\begin{aligned} v(2.5) &= 2\sin 2.5 + 1 \\ &= 2.20\ \mathrm{ms}^{-1} \end{aligned}$$

c

$$\begin{aligned} s(2\pi) &= \int_0^{2\pi} v(t)\ \mathrm{d}t \\ &\approx 6.28\ \mathrm{m} \end{aligned}$$

d

$$\begin{aligned} d(2\pi) &= \int_0^{2\pi} |v(t)|\ \mathrm{d}t \\ &\approx 9.02\ \mathrm{m} \end{aligned}$$

15 $v = s\mathrm{e}^{-s}$

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(s\mathrm{e}^{-s}) \\ &= \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{e}^{-s} - s\mathrm{e}^{-s}\frac{\mathrm{d}s}{\mathrm{d}t} \\ &= v\mathrm{e}^{-s}(1-s) \\ &= s\mathrm{e}^{-2s}(1-s) \end{aligned}$$

16 Implicit differentiation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(v^2) &= \frac{\mathrm{d}}{\mathrm{d}t}(4s^2+1) \\ 2v\frac{\mathrm{d}v}{\mathrm{d}t} &= 8s\frac{\mathrm{d}s}{\mathrm{d}t} \\ 2va &= 8sv \\ a &= 4s \end{aligned}$$

17

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{1}{4}t^4 + \frac{1}{2}t^3 - \frac{1}{10}t^2 + 4t$$

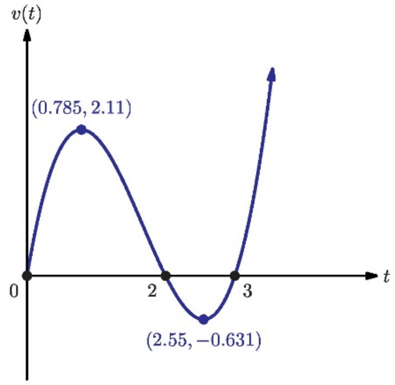
From GDC, maximum $|v(t)|$ is $|v(4)| \approx 17.6\ \mathrm{m\ s}^{-1}$

18 a Since graph C is linear, which is clearly the lowest degree of polynomial, this must be the acceleration, then the velocity is quadratic and the displacement cubic.

Function	Graph
displacement	B
acceleration	A

b Maximum velocity is at $t = 3$

19 a



b Velocity is increasing for $0 \leqslant -t < 0.785$ and $t > 2.55$

c Magnitude of the velocity is increasing for $0 \leqslant -t < 0.785, 2 < t < 2.55, t > 3$

d

$$x_A = \int v_A(t)\ \mathrm{d}t = \frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 + x_A(0)$$

$$x_A(0) = 0$$

$$x_A = \frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2$$

e $v_B = -20\mathrm{e}^{-2t}$

f

$$x_B = \int v_B(t)\ \mathrm{d}t = 10\mathrm{e}^{-2t} + c$$

$$x_B(0) = 20 = 10\mathrm{e}^{-2t} + c$$

$$c = 10$$

Intersection of the two paths: $\frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2 = 10 + 10\mathrm{e}^{-2t}$

From GDC, $t = 4.41$ seconds.

20 a $\mathbf{v_1} = \begin{pmatrix} 4t-2 \\ 5-2t \end{pmatrix}$ and $\mathbf{v_2} = \begin{pmatrix} \frac{8}{3}t \\ 2t-1 \end{pmatrix}$

$$\mathbf{r_1} = \int \mathbf{v_1}(t)\ \mathrm{d}t = \begin{pmatrix} 2t^2-2t \\ 5t-t^2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\mathbf{r_1}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\mathbf{r_1} = \begin{pmatrix} 2t^2-2t \\ 5t-t^2 \end{pmatrix}$$

$$\mathbf{r_2} = \int \mathbf{v_2}(t)\ \mathrm{d}t = \begin{pmatrix} \frac{4}{3}t^2 \\ t^2-t \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\mathbf{r_2}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\mathbf{r_2} = \begin{pmatrix} \frac{4}{3}t^2 \\ t^2-t \end{pmatrix}$$

Solving $\mathbf{r_1}(t) = \mathbf{r_2}(t)$

$$\begin{pmatrix} 2t^2-2t \\ 5t-t^2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}t^2 \\ t^2-t \end{pmatrix}$$

$$t\begin{pmatrix} 2t-2-\frac{4}{3}t \\ 5-t-t+1 \end{pmatrix} = 0$$

$$t\begin{pmatrix} \frac{2}{3}t-2 \\ 6-2t \end{pmatrix} = 0$$

Solution at $t = 0$ and $t = 3$, so the particles meet again at $t = 4$

b $\mathbf{v_1}(3) = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$, $\mathbf{v_2}(3) = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

Angle of $\mathbf{v_1}(3) = \arctan\left(-\frac{1}{10}\right) = -5.7°$

Angle of $\mathbf{v_2}(3) = \arctan\left(\frac{5}{8}\right) = 32.0°$

The angle between the velocities is 37.7°

21 a $a_1 = -9.8$

$$v_1 = \int a_1\ \mathrm{d}t = -9.8t + k$$

$$v_1(0) = 8 = k$$

$$v_1 = 8 - 9.8t$$

$$s_1 = \int v_1\ \mathrm{d}t = 8t - 4.9t^2 + c$$

$$s_1(0) = 0 = c$$

$$s_1 = 8t - 4.9t^2$$

b $s_2(t) = s_1(t-0.7) = 8(t-0.7) - 4.9(t-0.7)^2$

Using GDC to solve for $s_1(t) = s_2(t)$:

$t = 1.17$ seconds

22 a $\mathbf{v}(0) = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$ and $\mathbf{a}(t) = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$

$$\mathbf{v}(t) = \int \mathbf{a}(t)\ \mathrm{d}t = \begin{pmatrix} k_1 \\ k_2 - 9.8t \end{pmatrix}$$

$$\mathbf{v}(0) = \begin{pmatrix} 8 \\ 5 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$\mathbf{v}(t) = \begin{pmatrix} 8 \\ 5 - 9.8t \end{pmatrix}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 8t + c_1 \\ 5t - 4.9t^2 + c_2 \end{pmatrix}$$

$$\mathbf{r}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\mathbf{r}(t) = \begin{pmatrix} 8t \\ 5t - 4.9t^2 \end{pmatrix}$$

b Taking the time reference as zero at the time of the second launch,

$$\mathbf{v_2}(t) = \mathbf{v}(t) = \begin{pmatrix} 8 \\ 5 - 9.8t \end{pmatrix}, \mathbf{v_1}(\mathrm{t}) = \mathbf{v}(t + 0.5) = \begin{pmatrix} 8 \\ 5 - 9.8(t + 0.5) \end{pmatrix}$$

The horizontal element is constant and the same for both, so speeds will match when

$$(5 - 9.8t)^2 = \big(5 - 9.8(t + 0.5)\big)^2$$

From GDC, this has solution $t = 0.260$ seconds.

23 a

$$\mathbf{r_1}(t) = \int \mathbf{v}(t)\ \mathrm{d}t = \begin{pmatrix} 8\sin 2t + k_x \\ -8\cos 2t + k_y \end{pmatrix}$$

$$\mathbf{r_1}(0) = \begin{pmatrix} 0 \\ -8 \end{pmatrix} = \begin{pmatrix} k_x \\ k_y - 8 \end{pmatrix} \text{ so } \mathbf{r_1}(t) = \begin{pmatrix} 8\sin 2t \\ -8\cos 2t \end{pmatrix}$$

b $\mathbf{r_2}(t) = \mathbf{r_1}(t - 1.2)$

Require y-components to be the same:

$-8\cos(2t) = -8\cos(2t - 2.4)$

By symmetry of the cosine function, this will occur for

$2t = n\pi + 1.2$

so will occur first for positive t at $t = 0.5\pi + 0.6 \approx 2.17$ seconds.

$\mathbf{r}_1(0.5\pi + 0.6) = \begin{pmatrix} -7.45 \\ 2.90 \end{pmatrix}$ and $\mathbf{r}_2(0.5\pi + 0.6) = \mathbf{r}_1(0.5\pi - 0.6) = \begin{pmatrix} 7.45 \\ 2.90 \end{pmatrix}$

This means the particles are $8 + 2.90 = 10.9$ cm above the base of the circle.

24 a **i** $-10\ \mathrm{m\ s^{-2}}$

ii $v(10) = -100\ \mathrm{m\ s^{-1}}$

iii

$$s(t) = \int v\ \mathrm{d}t = -5t^2 + k$$

$$s(0) = 1000$$

$$s(t) = 1000 - 5t^2$$

So $s(10) = 500$

24 b

$$\frac{dv}{dt} = a(t) = -10 - 5v, t \geqslant 10$$
$$\frac{dt}{dv} = -\frac{1}{10 + 5v}$$

c Separating variables:

$$\int 1\,\mathrm{d}t = -\int \frac{1}{10+5v}\,\mathrm{d}v = -\frac{1}{5}\int \frac{1}{2+v}\,\mathrm{d}v$$
$$t = -\frac{1}{5}\ln|2+v| + c$$

When $t = 10, v = -100$

$$10 = -\frac{1}{5}\ln 98 + c$$
$$c = 10 + \frac{1}{5}\ln 98$$
$$t = 10 + \frac{1}{5}(\ln 98 - \ln|2+v|) = 10 + \frac{1}{5}\ln\left(\frac{98}{|2+v|}\right)$$

Since $a(t) = -5(2+v) \geqslant 0$, it follows that $2 + v \leqslant 0$ so $|2+v| = -2 - v$

$$t = 10 + \frac{1}{5}\ln\left(\frac{98}{-2-v}\right) \text{ for } t \geqslant 10$$

d Rearranging:

$$5(t-10) = -\ln\left(\frac{-2-v}{98}\right)$$
$$v = -98\mathrm{e}^{50-5t} - 2, \text{ for } t \geqslant 10$$

e

$$s = \int v\ \mathrm{d}t = \frac{98}{5}\mathrm{e}^{50-5t} - 2t + d, \text{ for } t \geq 10$$
$$s(10) = 500 = \frac{98}{5} - 20 + d$$
$$d = 520 - \frac{98}{5} = 500.4$$
$$s = \int v\ \mathrm{d}t = 500.4 + \frac{98}{5}\mathrm{e}^{50-5t} - 2t$$

f Solving for $s = 0$ using GDC: $t = 250$ seconds.

13 Differential equations

These are worked solutions to the colour-coded problem-solving questions from the exercises in the Student's Book. This excludes the drill questions.

Exercise 13A

Note: In many of these solutions, the initial unknown constant will be multiplied by a constant during rearrangement; a new letter is used for the amended constant without further justification, unless the transformation imposes limits upon it. For example, in Q14, the initial constant of integration k is multiplied by 3 to become c. Just as k is an arbitrary constant, so is $3k = c$, so no comment is required.

13

$$\int y^{-2}\ \mathrm{d}y = \int 1\ \mathrm{d}x$$
$$-y^{-1} = x + c$$
$$y = -\frac{1}{x+c}$$

14

$$\int y^2\ \mathrm{d}y = \int \cos x\ \mathrm{d}x$$
$$\frac{1}{3}y^3 = \sin x + k$$
$$y = \sqrt[3]{3\sin x + c}$$

15

$$\int \mathrm{e}^y\ \mathrm{d}y = \int 2x\ \mathrm{d}x$$
$$\mathrm{e}^y = x^2 + c$$
$$y = \ln(x^2 + c)$$

16

$$\int y^{-1}\ \mathrm{d}y = \int x^{-1}\ \mathrm{d}x$$
$$\ln|y| = \ln|x| + c$$
$$\ln\left|\frac{y}{x}\right| = c$$
$$\left|\frac{y}{x}\right| = \mathrm{e}^c = K$$

$K > 0$ is an arbitrary constant. Removing the modulus restriction,

$$\frac{y}{x} = A$$

where $A \neq 0$ is an arbitrary constant.

$y = Ax, A \neq 0$

Since $y = 0$ is also a solution of the differential equation, the full solution is $y = Ax$ with no restriction on constant A.

17

$$\int y^{-2}\ \mathrm{d}y = \int 2x\ \mathrm{d}x$$
$$-y^{-1} = x^2 + c$$
$$y = -\frac{1}{x^2 + c}$$

18 a

$$\int y^{-1}\ \mathrm{d}y = \int \sec^2 x\ \mathrm{d}x$$
$$\ln|y| = \tan x + c$$
$$|y| = \mathrm{e}^c\mathrm{e}^{\tan x} = A\mathrm{e}^{\tan x} \text{ for arbitrary constant } A > 0$$
$$y = A\mathrm{e}^{\tan x} \text{ for arbitrary constant } A$$

b

$$y(0) = 4 = A$$
$$y = 4\mathrm{e}^{\tan x}$$

19

$$\int 4y\ \mathrm{d}y = \int 9x^2\ \mathrm{d}x$$
$$2y^2 = 3x^3 + c$$
$$y(0) = 3$$
$$2(3)^2 = 3(0)^3 + c$$
$$c = 18$$
$$2y^2 = 3x^3 + 18$$

20

$$\int y^2 \ \mathrm{d}y = \int 3x \ \mathrm{d}x$$
$$\frac{1}{3}y^3 = \frac{3}{2}x^2 + c$$
$$y(2) = 3$$
$$\frac{1}{3}(3)^3 = \frac{3}{2}(2)^2 + c$$
$$9 = 6 + c$$
$$c = 3$$
$$y^3 = \left(\frac{9}{2}x^2 + 9\right)$$
$$y = \sqrt[3]{\frac{9}{2}(x^2 + 2)}$$

21 a

$$\int (y-1)^{-1} \ \mathrm{d}y = \int 2x + 4 \ \mathrm{d}x$$
$$\ln|y-1| = x^2 + 4x + c$$
$$|y-1| = \mathrm{e}^c\mathrm{e}^{x^2+4x} = A\mathrm{e}^{x^2+4x} \text{ for arbitrary constant } A > 0$$
$$y - 1 = A\mathrm{e}^{x^2+4x} \text{ for arbitrary } A$$
$$y = 1 + A\mathrm{e}^{x^2+4x}$$

b

$$y(0) = 2 = 1 + A \Rightarrow A = 1$$
$$y = 1 + \mathrm{e}^{x^2+4x}$$

22

$$\int \sec^2 y \ \mathrm{d}y = \int \cos x \ \mathrm{d}x$$
$$\tan y = \sin x + k$$
$$\sin x - \tan y = -k = c$$

23 a

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -km$$
$$\text{When } m = 25, \frac{\mathrm{d}m}{\mathrm{d}t} = -5 \text{ so } k = 0.2$$

b

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -0.2m$$
$$\int m^{-1} \ \mathrm{d}m = \int -0.2 \ \mathrm{d}t$$
$$\ln|m| = -0.2t + c$$
$$|m| = \mathrm{e}^c\mathrm{e}^{-0.2t} = A\mathrm{e}^{-0.2t} \text{ for arbitrary constant } A > 0$$
$$m = Ae^{-0.2t} \text{ for arbitrary constant } A$$
$$m(0) = 25 = A$$
$$m = 25\mathrm{e}^{-0.2t}$$

23 c When $m = 12.5, \mathrm{e}^{-0.2t} = 0.5$

$t = -5\ln 0.5 = 3.47$ seconds

24 a

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN$$

When $N = 2000, \frac{\mathrm{dN}}{dt} = 500$

$$k = 0.25$$

b

$$\int N^{-1}\ \mathrm{d}N = \int 0.25\ \mathrm{d}t$$

$$\ln|N| = 0.25t + c$$

$$|N| = \mathrm{e}^{c}\mathrm{e}^{0.25t} = A\mathrm{e}^{0.25t} \text{ for arbitrary constant } A > 0$$

$$N = A\mathrm{e}^{0.25t}$$

$$N(0) = A = 2000$$

$$N(10) = 2000\mathrm{e}^{2.5} \approx 24\,000$$

25 a

$$\frac{\mathrm{d}V}{\mathrm{d}t} = kV^{-1}$$

When $V = 300, \frac{\mathrm{d}V}{\mathrm{d}t} = 10$

$$\Rightarrow k = 10 \times 300$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{3000}{V}$$

b

$$\int V\ \mathrm{d}V = \int 3000\ \mathrm{d}t$$

$$\frac{1}{2}V^2 = 3000t + c$$

$$V = \sqrt{6000t + \tilde{c}}$$

$$V(0) = 300 = \sqrt{\tilde{c}}$$

$$\Rightarrow \tilde{c} = 90000$$

$$V(t) = \sqrt{6000t + 90000} = 20\sqrt{15(t + 15)}$$

26 a

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.1v = -0.1(v - 100)$$

Tip: When separating variables, it is often useful to factor out the expression so that the variable taken to the left has a coefficient of 1, and you leave the multiple on the right side. This saves additional rearrangement later.

$$\int (v - 100)^{-1}\,\mathrm{d}v = \int -0.1\ \mathrm{d}t$$

$$\ln|v - 100| = -0.1t + c$$

$$|v - 100| = \mathrm{e}^{c}\mathrm{e}^{-0.1t} = A\mathrm{e}^{-0.1t} \text{ for arbitrary constant } A > 0$$

$$v = 100 + A\mathrm{e}^{-0.1t}$$

$$v(0) = 0 = 100 + A$$

$$\Rightarrow A = -100$$

$$v = 100(1 - \mathrm{e}^{-0.1t})$$

b Velocity is always positive in this model, so distance travelled will be the integral of v over time:

$$d(3) = \int_0^3 v\ \mathrm{d}t = 40.8\text{ m}$$

27

$$\int y\ \mathrm{d}y = \int 4\mathrm{e}^{-2x}\ \mathrm{d}x$$

$$\frac{1}{2}y^2 = -2\mathrm{e}^{-2x} + c$$

$$y = \pm\sqrt{2c - 4\mathrm{e}^{-2x}}$$

$$y(0) = -2 = \pm\sqrt{2c - 4}$$

The curve follows the negative root for the model $2c - 4 = 4\ so\ c = 4$

$$y = -2\sqrt{2 - \mathrm{e}^{-2x}}$$

28

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x+y} = \mathrm{e}^{x}\mathrm{e}^{y}$$

$$\int \mathrm{e}^{-y}\ \mathrm{d}y = \int \mathrm{e}^{x}\ \mathrm{d}x$$

$$-\mathrm{e}^{-y} = \mathrm{e}^{x} + c$$

$$-y = \ln(-c - \mathrm{e}^{x})$$

$$y = -\ln(k - \mathrm{e}^{x})$$

29

$$\frac{dy}{dx} = 2e^{x-2y} = 2e^{x}e^{-2y}$$
$$\int e^{2y}\ dy = \int 2e^{x}\ dx$$
$$\frac{1}{2}e^{2y} = 2e^{x} + c$$
$$e^{2y} = 4e^{x} + k$$
$$y = \frac{1}{2}\ln(k + 4e^{x})$$
$$y(0) = 0 = \frac{1}{2}\ln(k + 4)\ \text{so}\ k = -3$$
$$y = \frac{1}{2}\ln(4e^{x} - 3)$$

30

$$\int y\ dy = \int \sin x\ dx$$
$$\frac{1}{2}y^{2} = c - \cos x$$
$$y = \pm\sqrt{2c - 2\cos x}$$
$$y(0) = 10 = \pm\sqrt{2c - 2}$$

so the curve follows the positive root and $2c = 102$

$$y = \sqrt{102 - 2\cos x}$$

31 a

$$\frac{dy}{dx} = \frac{\cos x}{\sin y}$$
$$\int \sin y\ dy = \int \cos x\ dx$$
$$-\cos y + c = \sin x$$
$$\sin x + \cos y = c$$

b

$$y\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$$
$$\sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right) = c = 1$$
$$\sin x + \cos y = 1$$

When $x = \frac{\pi}{2}, \sin x = 1$

$$\cos y = 0$$
$$y = \pm\frac{\pi}{2}$$

32 a $V = \frac{4}{3}\pi r^3, S = 4\pi r^2$

Implicit differentiation gives:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t} = S\frac{\mathrm{d}r}{\mathrm{d}t}$$

But $\frac{\mathrm{d}V}{\mathrm{d}t} = -kS$

So $\frac{\mathrm{d}r}{\mathrm{d}t} = -k$ (constant)

When $V = 0.5, \frac{\mathrm{d}V}{\mathrm{d}t} = -0.1$

$$\frac{4}{3}\pi r^3 = \frac{1}{2}$$

$$r = \sqrt[3]{\frac{3}{8\pi}}$$

$$-k(4\pi r^2) = -0.1$$

$$k = \frac{0.1}{4\pi r^2} = 0.0328$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -0.0328 \text{ cm s}^{-1}$$

b Integrating:

$$r = r(0) - kt$$

$$r(0) = \sqrt[3]{\frac{3}{8\pi}}$$

When $r = 0$

$$t = \frac{r(0)}{k}$$

$$= r(0) \div \left(\frac{0.1}{4\pi\big(r(0)\big)^2}\right)$$

$$= 40\pi\big(r(0)\big)^3$$

$$= 40\pi\left(\frac{3}{8\pi}\right)$$

$= 15$ minutes

Exercise 13B

Note: A table is shown in the working for Q11 so that students can check the detail of their calculator output.

A full table of results is not typically needed for this sort of problem unless specifically required in the examination question.

Lay out the basis by which you generate the values (the iteration formula) and then give the end results, indicating use of GDC.

11

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x = \mathrm{f}(x, y)$$
$$y_{n+1} = y_n + h \times \mathrm{f}(x_n, y_n)$$
$$x_0 = y_0 = 0$$

n	x	$y\ (h = 0.1)$	$y\ (h = 0.2)$
0	0	0	0
1	0.1	0	
2	0.2	0.02	0
3	0.3	0.06	
4	0.4	0.12	0.08
5	0.5	0.2	
6	0.6	0.3	0.24
7	0.7	0.42	
8	0.8	0.56	0.48
9	0.9	0.72	
10	1	0.9	0.8
11	1.1	1.1	
12	1.2	1.32	1.2
13	1.3	1.56	
14	1.4	1.82	1.68
15	1.5	2.1	
16	1.6	2.4	2.24
17	1.7	2.72	
18	1.8	3.06	2.88
19	1.9	3.42	
20	2	3.8	3.6

a For $h = 0.1$:

i $y(1) = 0.9$

ii $y(2) = 3.8$

b For $h = 0.2$:

i $y(1) = 0.8$

ii $y(2) = 3.6$

c

$$y = x^2 + c$$
$$y(0) = 0 = c$$
$$y = x^2$$

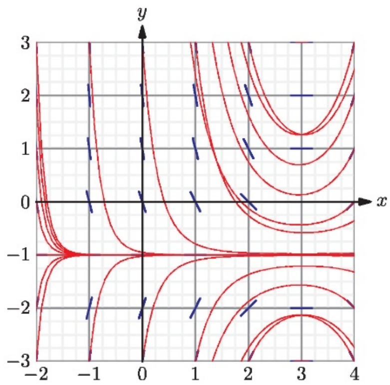
So, in absolute difference, **b ii** is the furthest from the true value.

Tip: Because a sketch of a slope field is always going to be imprecise, it is worthwhile to lay out a table of calculated slopes if the table gives a sensible restricted set of x, y values. If the table would be so large that such calculation would be a gross waste of time, just sketch the field.

Graphs in these worked solutions typically will show the complete field with solution curves drawn through every edge vertex. In practice you only need show a representative spread of curves to give a reasonable impression of the solution curve shapes.

12 a, b

		x						
		0	**1**	**2**	**3**	**4**	**5**	**6**
	3	−12	−8	−4	0	4	8	12
	2	−9	−6	−3	0	3	6	9
	1	−6	−4	−2	0	2	4	6
y	**0**	−3	−2	−1	0	1	2	3
	−1	0	0	0	0	0	0	0
	−2	3	2	1	0	−1	−2	−3
	−3	6	4	2	0	−2	−4	−6



c Any solution curve at $y = -1$ will be a constant (horizontal line); the system will be in equilibrium.

13

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{x+y} = \mathrm{f}(x,y)$$

$$y_{n+1} = y_n + h \times \mathrm{f}(x_n, y_n)$$

a From GDC: $y(10) \approx 31.4$

b Using a smaller step length would improve the estimate.

14

$$t_n = nh$$
$$x_{n+1} = x_n + h \times 0.005x(x - 3y - 20)$$
$$y_{n+1} = y_n + h \times 0.005y(3x - 5y - 300)$$

$h = 0.1, x_0 = 150, y_0 = 40$

From GDC: $x(0.5) \approx 157$ and $y(0.5) \approx 37$

After 6 months there are approximately 157 rabbits and 37 foxes.

15

$$t_n = nh$$
$$x_{n+1} = x_n + h \times -0.2x\cos(\pi t)$$
$$y_{n+1} = y_n + h \times (-0.5y + 0.2x)\cos(\pi t)$$

$h = 0.01, x_0 = 8.5, y_0 = 10.2$

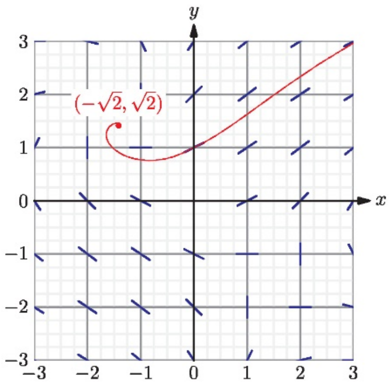
Three months equates to 0.25 years.

From GDC: $x(0.25) \approx 8.12, y(0.25) = 9.46$

The reservoirs are 8.12 m and 9.46 m deep.

16 a, b

		x				
		−2	**−1**	**0**	**1**	**2**
	2	0	∞	1	0.75	0.67
	1	∞	0	0.5	0.67	0.75
y	**0**	−1	−0.5	0	0.5	1
	−1	−0.75	−0.67	−0.5	0	∞
	−2	−0.67	−0.75	−1	∞	0



c From GDC:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+y)}{xy+2} = \mathrm{f}(x,y)$$

$$y_{n+1} = y_n + h \times \mathrm{f}(x_n, y_n)$$

$$x_0 = 0, y_0 = 1, h = 0.2$$

From GDC, when $x = 1$, $y \approx 1.629$

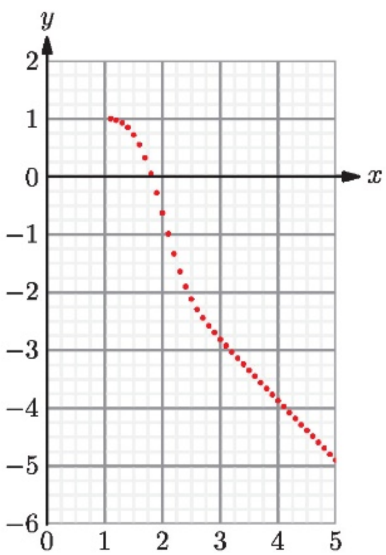
17

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2 = f(x,y)$$

$$y_{n+1} = y_n + h \times f(x_n, y_n)$$

$$x_0 = 1, y_0 = 1, h = 0.1$$

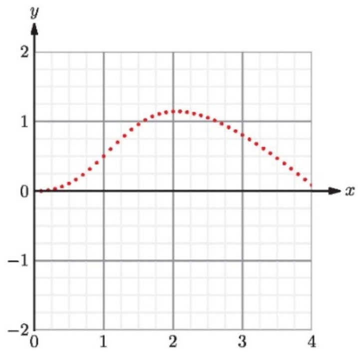
From GDC:



18 a

$$\frac{dy}{dx} = \sin(x + y) \ = f(x, y)$$
$$y_{n+1} = y_n + h \times f(x_n, y_n)$$
$$x_0 = 0, y_0 = 0, h = 0.1$$

From GDC:



b From the calculator data table, maximum y for $0 \leq x \leq 4$ is approximately 1.1.

19 Tip: Since the dependent variable in this question is h, we need a new letter for the Euler method step size. Greek letter delta (δ) is often used for small increments.

$$\frac{dh}{dt} = -0.1h^2 - 0.5t = f(h, t)$$
$$h_{n+1} = h_n + \delta \times f(t_n, h_n)$$
$$t_0 = 0, h_0 = 2, \delta = 0.1$$

a From GDC, $h(1) \approx 1.46$ m

b In the model using Euler's approximation, $h(2.5) \approx 0.05$ m, $h(2.6) \approx -0.08$ m.

It will take approximately 3 seconds for the ash to reach the fire.

20 a

$$\frac{dy}{dx} = xe^y \ = f(x, y)$$
$$y_{n+1} = y_n + h \times f(x_n, y_n)$$
$$x_0 = 1, y_0 = 0.3, h = 0.1$$

From GDC, when $x = 1.3, y \approx 0.825$

b

$$\int e^{-y} \ dy = \int x \ dx$$
$$-e^{-y} = \frac{1}{2}x^2 + c$$

When $y = 0.3, x = 1$:

$$-e^{-0.3} = \frac{1}{2} + c \Rightarrow c = -\frac{1}{2} - e^{-0.3}$$
$$e^{-y} = \frac{1}{2}(1 - x^2) + e^{-0.3}$$
$$y = -\ln\left(\frac{1}{2}(1 - x^2) + e^{-0.3}\right)$$

20 c i

$$\begin{aligned}\text{Percentage error} &= \frac{|\text{approximate value} - \text{true value}|}{|\text{true value}|} \times 100\% \\ &= \frac{\left|0.825 + \ln\left(\frac{1}{2}(1 - 1.3^2) + e^{-0.3}\right)\right|}{\left|\ln\left(\frac{1}{2}(1 - 1.3^2) + e^{-0.3}\right)\right|} \times 100\% \\ &= 11.0\%\end{aligned}$$

ii Taking a smaller increment would be expected to reduce the error in the approximation.

21 a i From GDC: $f(1) \approx 0.615$

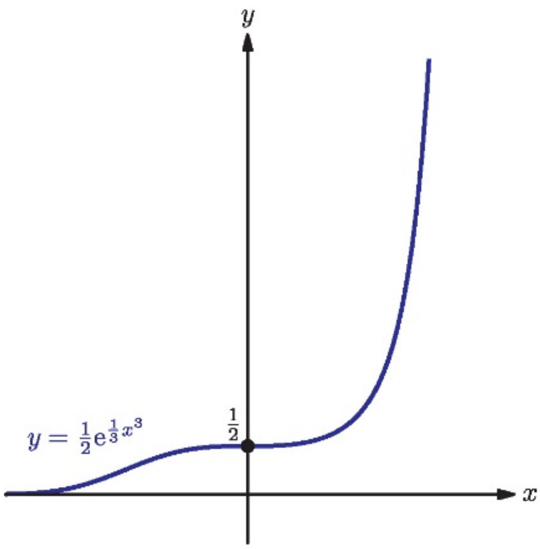
ii Accuracy would be expected to increase for a smaller step length.

b

$$\begin{aligned}\int y^{-1}\ \mathrm{d}y &= \int x^2\ \mathrm{d}x \\ \ln|y| &= \frac{1}{3}x^3 + c \\ |y| &= A\mathrm{e}^{\frac{1}{3}x^3} \\ y(0) &= 0.5 = A \\ y &= \frac{1}{2}\mathrm{e}^{\frac{1}{3}x^3}\end{aligned}$$

Then $y(1) = 0.698$

c



The curve is concave-up for $x > 0$; in consequence, the tangent always lies below the curve, so the Euler approximation will always underestimate.

22 Calculating distance cumulatively, with

$d_{n+1} = d_n + (t_{n+1} - t_n)v_n$

n	0	1	2	3	4	5
t_n	0	3	6	9	12	15
v_n	0	6	12	19	24	27
d_n	0	0	18	54	111	183

Estimate distance travelled in the first 15 seconds as 183 m

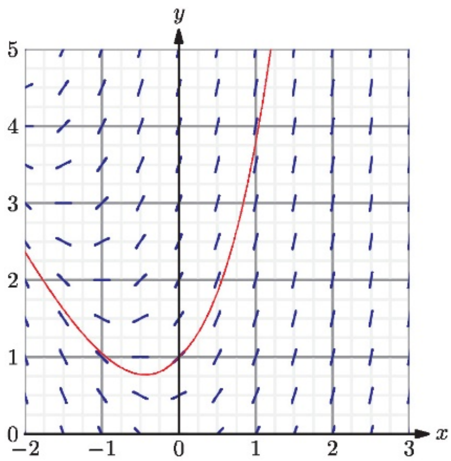
23 Calculating distance cumulatively, with

$d_{n+1} = d_n + (t_{n+1} - t_n)v_n$

n	0	1	2	3	4	5
t_n	0	3	6	9	12	15
v_n	20	15	10	8	6	5
d_n	0	60	105	135	159	177

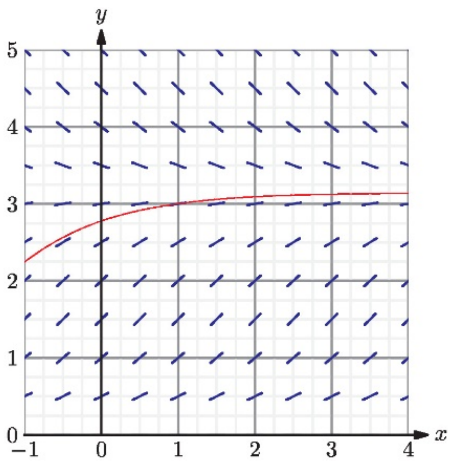
Estimate distance travelled in the first 15 seconds as 177 m

24 a



b $y(1) \approx 3.8$

25 a



b $y(3) \approx 3.1$

Exercise 13C

13 a $\det\begin{pmatrix}-\lambda & 2\\ 8 & 1-\lambda\end{pmatrix} = 0$

$$(1-\lambda)^2 - 16 = 0$$
$$\lambda = -3 \text{ or } 5$$

When $\lambda = -3$:

$$\begin{pmatrix}x+2y\\ 8x+y\end{pmatrix} = -3\begin{pmatrix}x\\ y\end{pmatrix}$$
$$4x = -2y$$
$$\mathbf{p_1} = \begin{pmatrix}1\\ -2\end{pmatrix}$$

When $\lambda = 5$:

$$\begin{pmatrix}x+2y\\ 8x+y\end{pmatrix} = 5\begin{pmatrix}x\\ y\end{pmatrix}$$
$$4x = 2y$$
$$\mathbf{p_2} = \begin{pmatrix}1\\ 2\end{pmatrix}$$

b $\begin{pmatrix}x\\ y\end{pmatrix} = A\mathrm{e}^{-3t}\begin{pmatrix}1\\ -2\end{pmatrix} + B\mathrm{e}^{5t}\begin{pmatrix}1\\ 2\end{pmatrix}$

c Substituting $x(0) = y(0) = 4$:

$$\begin{pmatrix}4\\ 4\end{pmatrix} = \begin{pmatrix}A+B\\ -2A+2B\end{pmatrix}$$

$A = 1, B = 3$

$$\begin{cases}x = \mathrm{e}^{-3t} + 3\mathrm{e}^{5t}\\ y = 6\mathrm{e}^{5t} - 2\mathrm{e}^{-3t}\end{cases}$$

14 a $\det\begin{pmatrix}1-\lambda & 3\\ 2 & 2-\lambda\end{pmatrix} = 0$

$$2 - 3\lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$
$$\lambda = 4 \text{ or } -1$$

b

$$x = A\mathrm{e}^{-t} + B\mathrm{e}^{4t}$$
$$x' = 4B\mathrm{e}^{4t} - A\mathrm{e}^{-t}$$

$x(0) = 4 = A + B$ (1)

$x'(0) = 1 = 4B - A$ (2)

$(1) + (2) \Rightarrow 5B = 5$

$B = 1, A = 3$

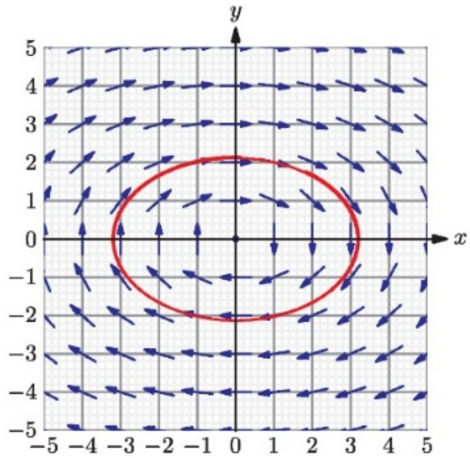
$x = 3\mathrm{e}^{-1} + \mathrm{e}^{4t}$

15 a $\det\begin{pmatrix}-\lambda & 0.9\\ -0.4 & -\lambda\end{pmatrix} = 0$

$$\lambda^2 + 0.36 = 0$$
$$\lambda = \pm 0.6\mathrm{i}$$

15 b i When $y = 2$ $\dot{x} = 1.8 > 0$ so x is increasing.

ii



16 a $\det\begin{pmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{pmatrix} = 0$

$$12 - 7\lambda + \lambda^2 - 6 = 0$$
$$\lambda^2 - 7\lambda + 6 = 0$$
$$\lambda = 1 \text{ or } 6$$

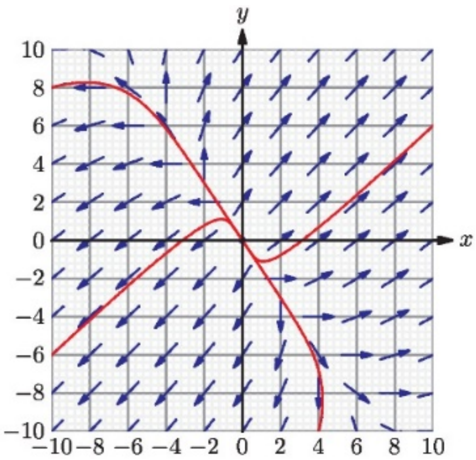
When $\lambda = 1$:

$$\begin{pmatrix} 4x + 2y \\ 3x + 3y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$3x = -2y$$
$$\mathbf{p_1} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

When $\lambda = 6$:

$$\begin{pmatrix} 4x + 2y \\ 3x + 3y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix}$$
$$2x = 2y$$
$$\mathbf{p_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b



16 c $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{t}\begin{pmatrix} 2 \\ -3 \end{pmatrix} + B\mathrm{e}^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x(0) = 6, y(0) = 1$

$2A + B = 6 \qquad (1)$

$-3A + B = 1 \qquad (2)$

$(1) - (2) \Rightarrow 5A = 5$

$A = 1, B = 4$

$$\begin{cases} x = 2\mathrm{e}^{t} + 4\mathrm{e}^{6t} \\ y = -3\mathrm{e}^{t} + 4\mathrm{e}^{6t} \end{cases}$$

d The model is not suitable for long term, since it predicts unlimited population growth for both fish and algae.

17 a Maximum y appears to be approximately 3.7.

b The model appears to have a stable equilibrium at $x = 2, y = 2$.

18 a Not true; x decreases for values in the lower right part of the portrait diagram.

b True

c True

d Not true; y moves towards 6 from below and from above,
so values of y greater than 6 will decrease.

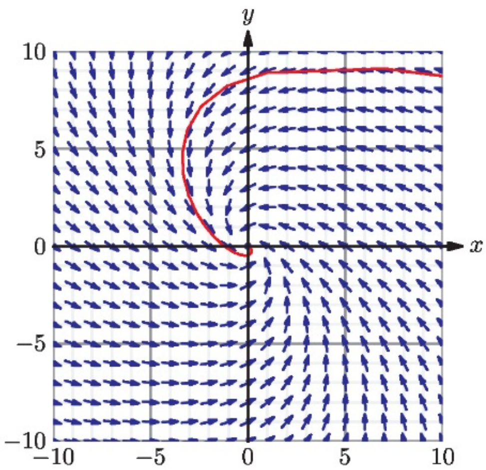
19 a $\det\begin{pmatrix} -3-\lambda & -2 \\ 2 & -1-\lambda \end{pmatrix} = 0$

$3 + 4\lambda + \lambda^2 + 4 = 0$

$\lambda^2 + 4\lambda + 7 = 0$

$\lambda = -2 \pm \mathrm{i}\sqrt{3}$

At $(4, 9)$, x is decreasing and y is also decreasing.



Note: From the eigenvalues you know that both variables have a function of the form

$\mathrm{e}^{-2t}\left(A\cos\sqrt{3}t + B\sin\sqrt{3}\,t\right)$

The specifics are not needed. In this note, you can see how you might find the exact coefficients but this is not needed for this question; the general form is enough to predict the behaviour, given you know the start values for each variable.

$$x = e^{-2t}\left(A\cos\sqrt{3}t + B\sin\sqrt{3}\,t\right)$$
$$\dot{x} = -3x - 2y$$
$$y = -\frac{1}{2}(\dot{x} + 3x) = -\frac{1}{2}e^{-2t}\left(\left(A + \sqrt{3}B\right)\cos\sqrt{3}t + \left(B - \sqrt{3}A\right)\sin\sqrt{3}t\right)$$
$$x(0) = 4: A = 4$$
$$y(0) = 9: -\frac{\sqrt{3}B}{2} - 2 = 9 \text{ so } B = -\frac{22\sqrt{3}}{3}$$
$$\begin{cases} x = e^{-2t}\left(4\cos\sqrt{3}t - \dfrac{22\sqrt{3}}{3}\sin\sqrt{3}t\right) \\ y = e^{-2t}\left(9\cos\sqrt{3}t - \dfrac{34\sqrt{3}}{3}\sin\sqrt{3}t\right) \end{cases}$$

19 b

c Both x and y oscillate with decreasing amplitude towards a stable equilibrium at $x = 0, y = 0$.

20 a $\det\begin{pmatrix} -8-\lambda & 6 \\ -9 & 13-\lambda \end{pmatrix} = 0$

$$-104 - 5\lambda + \lambda^2 + 54 = 0$$
$$\lambda^2 - 5\lambda - 50 = 0$$
$$(\lambda - 10)(\lambda + 5) = 0$$
$$\lambda = 10 \text{ or } -5$$

When $\lambda = 10$:

$$\begin{pmatrix} -8x + 6y \\ -9x + 13y \end{pmatrix} = 10\begin{pmatrix} x \\ y \end{pmatrix}$$
$$18x = 6y$$
$$\mathbf{p_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

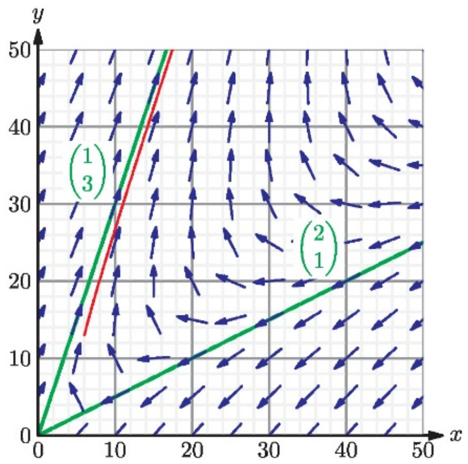
When $\lambda = -5$:

$$\begin{pmatrix} -8x + 6y \\ -9x + 13y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$$
$$3x = 6y$$
$$\mathbf{p_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

20 b

$$x(t) = Ae^{10t} + Be^{-5t}$$
$$x(0) = 6 = A + B \qquad (1)$$
$$y = \frac{1}{6}(\dot{x} + 8x) = \frac{1}{6}(18Ae^{10t} + 3Be^{-5t})$$
$$y(0) = 13 = 3A + \frac{B}{2} \qquad (2)$$
$$2(2) - (1) \Rightarrow 5A = 20$$
$$A = 4, B = 6 - A = 2$$
$$\begin{cases} x = 4e^{10t} + 2e^{-5t} \\ y = 12e^{t} + e^{-5t} \end{cases}$$

c



d In the long term, the e^{-5t} terms decay to insignificance, and the end behaviour is

$x = 4e^{10t}, y = 12e^{10t}$

so $y = 3x$

The ratio of foxes to rabbits tends to 1: 3.

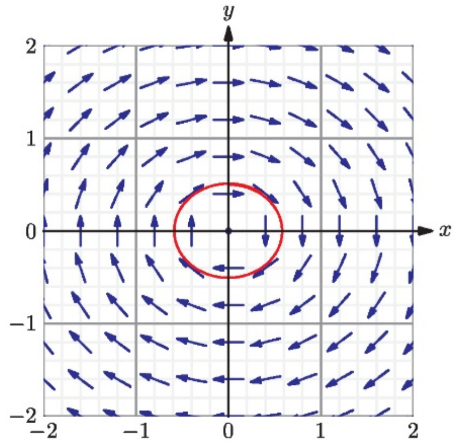
21 a $\det\begin{pmatrix} -\lambda & 0.8 \\ -0.6 & -\lambda \end{pmatrix} = 0$

$$\lambda^2 + \frac{12}{25} = 0$$
$$\lambda = \pm\frac{2\sqrt{3}}{5}\mathrm{i}$$

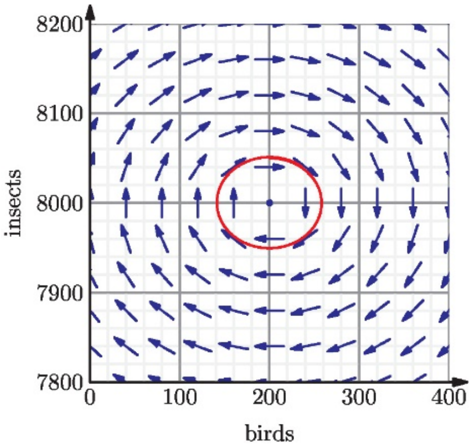
B 200 birds and 8050 insects.

At $y = 0.5$, $\dot{x} > 0$ so the bird population is increasing.

21 c Eigenvalues $a \pm b\mathrm{i}$ for $a = 0$ will show stable cycles.

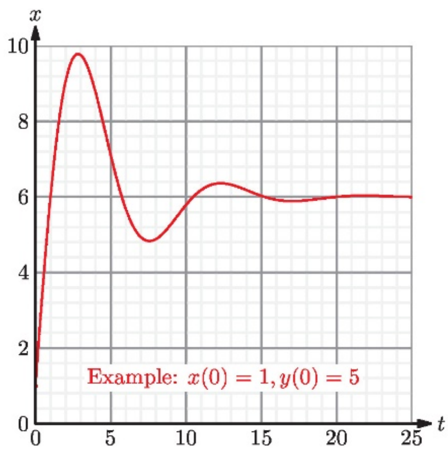


d The populations will oscillate around the values $x = 0, y = 0$ (200 birds, 8000 insects).



22 a The number of spiders tends to 600 and the number of flies tends to 300.

b



23 a The long term trajectories all seem to have x staying fairly constant for increasing y.

b The origin appears to be a saddle point, with the inward (negative eigenvalue) eigenvector being approximately $(200,20)$.

This would accord with option **iii**, where eigenvalue -0.3 is associated with eigenvector $\begin{pmatrix} 10 \\ 1 \end{pmatrix}$. The outward (positive eigenvalue) eigenvector has y increase dominating, so eigenvalue 3 with eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ would also be consistent.

24 a $\dot{x} = \dot{y} = 0$:

$$\begin{cases} -3x + 4y = 14 & (1) \\ -2x + y = 1 & (2) \end{cases}$$

$(1) - 4(2) \Rightarrow 5x = 10$

$x = 2, y = 5$

The equilibrium populations are 200 sharks and 500 fish.

b $u = x - 2$ and $v = y - 5$, so $x = u + 2, y = v + 5$

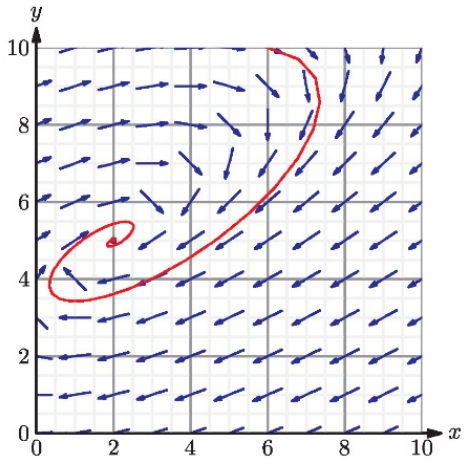
$$\dot{u} = \dot{x} = -3(u + 2) + 4(v + 5) - 14 = -3u + 4v$$
$$\dot{v} = \dot{y} = -2(u + 2) + (v + 5) - 1 = -2u + v$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

c $\det \begin{pmatrix} -3 - \lambda & 4 \\ -2 & 1 - \lambda \end{pmatrix} = 0$

$$-3 + 2\lambda + \lambda^2 + 8 = 0$$
$$\lambda^2 + 2\lambda + 5 = 0$$
$$\lambda = -1 \pm 2\text{i}$$

d Eigenvalues $a \pm b\text{i}$ with $a < 0$ causes a trajectory following an inward spiral.



e The number of sharks oscillates around and tends to 200;
the number of fish oscillates around and tends to 500.

Note: You can see on the phase plane diagram in the solution that the model has some anomalous behaviour; it would allow populations of fish or sharks to go negative. More sophisticated models, even for interacting populations, will often show for any population x that $\dot{x}$ has x as a factor, so that $x = 0$ becomes an asymptote and trajectories cannot pass through it.

Try using technology to adapt the models to

$$\dot{x} = x(-3x + 4y - 14)$$

$$\dot{y} = y(-2x + y - 1)$$

The equilibrium remains at the same value, but the origin also becomes an equilibrium point, and you should see more reasonable behaviours when either sharks or fish populations are very small.

Exercise 13D

Although it is tedious doing so repeatedly in an exercise where all questions have the same premise, it is good to practice always giving the Euler method as part of your answer. Since most of the calculation is done on a calculator, you can really only show working by writing explicitly the recurrence relation you will be using.

9 **a**

$$\text{If } y = \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$\text{then } \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + 2x = y + 2x$$

b Euler's method:

$$x(t+h) = x(t) + h \times \frac{\mathrm{d}x}{\mathrm{d}t}(t)$$
$$y(t+h) = y(t) + h \times \frac{\mathrm{d}y}{\mathrm{d}t}(t)$$

i From GDC: $h = 0.5$ gives $x(10) \approx 699000$

ii From GDC: $h = 1.0$ gives $x(10) \approx 39400$

c Answer **bi** can be expected to be more accurate because the step length is smaller.

10 **a**

$$\text{If } y = \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$\text{then } \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - x + 1 - t = y - x + 1 - t$$

b Euler's method:

$$x(t+h) = x(t) + h \times \frac{\mathrm{d}x}{\mathrm{d}t}(t)$$
$$y(t+h) = y(t) + h \times \frac{\mathrm{d}y}{\mathrm{d}t}(t)$$

From GDC: $x(0) = 0, y(0) = 2, h = 0.2$ gives $x(4) \approx -3.46$

11 **a**

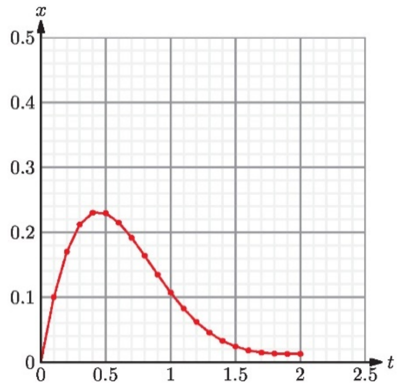
$$\text{If } y = \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$\text{then } \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -4\frac{\mathrm{d}x}{\mathrm{d}t} - 9x + \mathrm{e}^{-\mathrm{t}}$$
$$= -4y - 9x + \mathrm{e}^{-t}$$

b Euler's method:

$$x(t+h) = x(t) + h \times \frac{\mathrm{d}x}{\mathrm{d}t}(t)$$
$$y(t+h) = y(t) + h \times \frac{\mathrm{d}y}{\mathrm{d}t}(t)$$

From GDC: $x(0) = 0, y(0) = 1, h = 0.1$ gives $x(1) \approx 0.107$

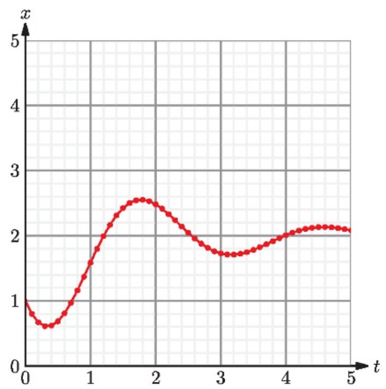
c



d From the graph, the maximum displacement is approximately 0.23.

12 Euler's method:

$$p(t+h) = p(t) + h \times \dot{p}(t)$$
$$\dot{p}(t+h) = \dot{p}(t) + h \times \ddot{p}(t)$$
$$\ddot{p}(t) = -1.5\dot{p}(t) - 5p + 10 - \mathrm{e}^{-t}$$
$$p(0) = 1, \dot{p}(0) = 2, h = 0.1$$



a Purchase at about 0.3 years, when the stock price is at its lowest.

b Sell at 1.8 years, when the stock price is highest, so keep the shares for 1.5 years.

c Purchase price at 0.3 years is approximately \$0.61.

Sale price at 1.8 years is approximately \$2.55.

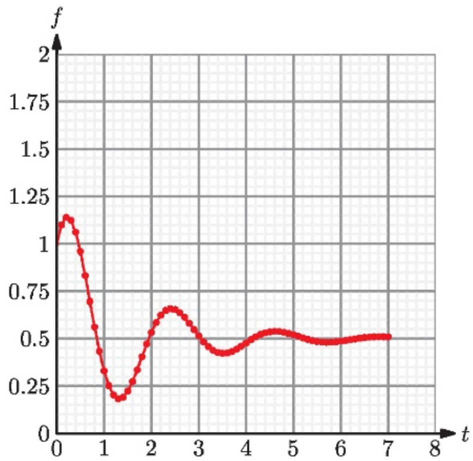
Profit per share is approximately \$1.94.

d The long term price of the shares is approximately \$2.00.

e The model predicts complete stability in the long term, which is unlikely given the behaviour of share prices in general and their reactivity to other factors.

13 a Euler's method:

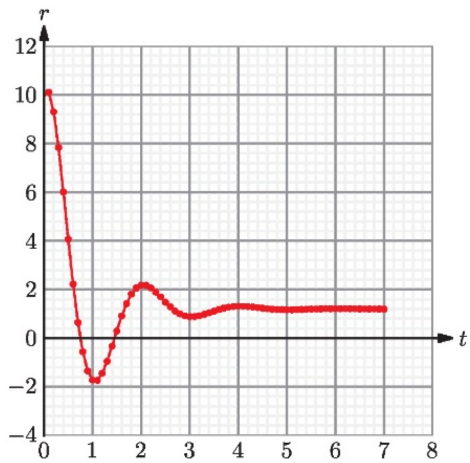
$$f(t+h) = f(t) + h \times \dot{f}(t)$$
$$\dot{f}(t+h) = \dot{f}(t) + h \times \ddot{f}(t)$$
$$\ddot{f}(t) = -2\dot{f}(t) - 8f + 4$$
$$f(0) = 1, \dot{f}(0) = 1, h = 0.1$$



The population f oscillates with decreasing amplitude, tending to a stable population at $f = 0.5$ which represents 500 foxes. ($f = 0.5, \dot{f} = \ddot{f} = 0$ is a stable equilibrium)

b Euler's method:

$$r(t+h) = r(t) + h \times \dot{r}(t)$$
$$\dot{r}(t+h) = \dot{r}(t) + h \times \ddot{r}(t)$$
$$\ddot{r}(t) = -3\dot{r}(t) - 10r + 12$$
$$r(0) = 10, \dot{r}(0) = 1, h = 0.1$$

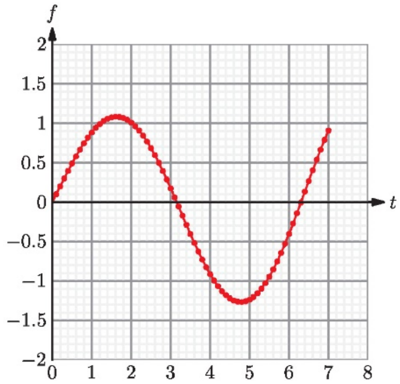


The model suggests that the rabbit population dips into the negative numbers, which is clearly impossible.

14 a Euler's method:

$$x(t+h) = x(t) + h \times \dot{x}(t)$$
$$\dot{x}(t+h) = \dot{x}(t) + h \times \ddot{x}(t)$$
$$\ddot{x}(t) = -x$$
$$x(0) = 0, \dot{x}(0) = 1, h = 0.1$$

14 a i



ii Amplitude is approximately 1.1 mm and the period is approximately 6.3 ms.

b i $\ddot{x}(t) = -x - \sin t$

$x(0) = 0, \dot{x}(0) = 0, h = 0.1$

From GDC, maximum displacement for $0 \leqslant t \leqslant 7$ is 3.7 mm.

ii $\ddot{x}(t) = -x - \sin 4t$

$x(0) = 0, \dot{x}(0) = 0, h = 0.1$

From GDC, maximum displacement for $0 \leqslant t \leqslant 7$ is 0.39 mm.

15 Euler's method:

$$y(t+h) = y(t) + h \times \dot{y}(t)$$
$$\dot{y}(t+h) = \dot{y}(t) + h \times \ddot{y}(t)$$
$$\ddot{y}(t) = 0.7\dot{y}(t) - 0.12y(t)$$

a i $y(0) = 1, \dot{y}(0) = 0.2, h = 0.5$

From GDC: $y(3) \approx 1.64$

ii $y(0) = 1, \dot{y}(0) = 0.2, h = 0.1$

From GDC: $y(3) \approx 1.61$

b Let $z = \dot{y}$

Then $\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0y + z \\ -0.12y + 0.7z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -0.12 & 0.7 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$

Eigenvalues λ are given by $\det \begin{pmatrix} -\lambda & 1 \\ -0.12 & 0.7 - \lambda \end{pmatrix} = 0$

$\lambda^2 - 0.7\lambda + 0.12 = 0$

$\lambda = 0.3$ or 0.4

Then $y = Ae^{0.3t} + Be^{0.4t}$

So $\dot{y} = 0.3Ae^{0.3t} + 0.4Be^{0.4t}$

$y(0) = 1 = A + B \qquad (1)$

$\dot{y}(0) = 0.2 = 0.3A + 0.4B \qquad (2)$

$4(1) - 10(2) \Rightarrow A = 2$ so $B = -1$

$y = 2e^{0.3t} - e^{0.4t}$

15 c From part **b**, $y(3) = 1.599$

$$\text{Percentage error} = \frac{|\text{estimated value} - \text{actual value}|}{|\text{actual value}|} \times 100\%$$

i Percentage error $= 2.57\%$

ii Percentage error $= 0.753\%$

16 a Euler's method:

$$y(t+h) = y(t) + h \times \dot{y}(t)$$
$$\dot{y}(t+h) = \dot{y}(t) + h \times \ddot{y}(t)$$
$$\ddot{y}(t) = -(\dot{y}(t))^2 - (y(t))^2 + 1 - \sin t$$
$$y(0) = 0, \dot{y}(0) = 1, h = 0.1$$

i From GDC: $y(1) \approx 0.875$

ii From GDC: $y(2) \approx 0.965$

b If $y = \sin t$ then $\dot{y} = \cos t$ and $\ddot{y} = -\sin t$, then

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2 + y^2 = -\sin t + \cos^2 t + \sin^2 t = 1 - \sin t$$

Also, $y(0) = \sin 0 = 0$ and $\dot{y}(0) = \cos 0 = 1$

So $y = \sin t$ satisfies the differential equation and the initial conditions.

c

$$\text{Percentage error} = \frac{|\text{estimated value} - \text{actual value}|}{|\text{actual value}|} \times 100\%$$

i $y(1) = 0.841$ so Percentage error $= 4.02\%$

ii $y(1) = 0.909$ so Percentage error $= 6.12\%$

Mixed Practice

1 a

$$\int y^{-1}\ \mathrm{d}y = \int 3\cos 2x\ \mathrm{d}x$$
$$\ln|y| = 1.5\sin 2x + c$$
$$|y| = A\mathrm{e}^{1.5\sin 2x} \text{ for } A > 0$$
$$y = A\mathrm{e}^{1.5\sin 2x}$$

b $y(0) = 5 = A$

$$y = 5\mathrm{e}^{1.5\sin 2x}$$

2

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x^2 - 2)y$$
$$\int y^{-1}\ \mathrm{d}y = \int 3x^2 - 2\ \mathrm{d}x$$
$$\ln|y| = x^3 - 2x + c$$
$$|y| = A\mathrm{e}^{x^3 - 2x} \text{ for } A > 0$$
$$y = A\mathrm{e}^{x^3 - 2x}$$

3 Euler's method:

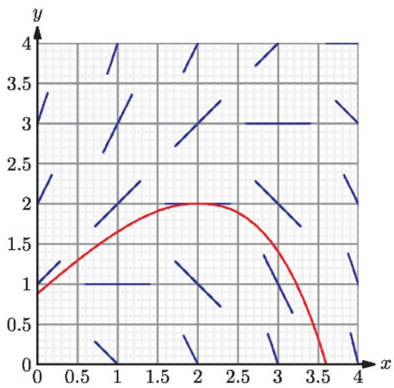
$$y(x+h) = y(x) + h \times \frac{\mathrm{d}y}{\mathrm{d}x}(x)$$

$$y(0) = 2, h = 0.1$$

From GDC: $y(2) \approx 2.45$

4 **a, b**

		x				
		0	**1**	**2**	**3**	**4**
y	**4**	4	3	2	1	0
	3	3	2	1	0	−1
	2	2	1	0	−1	−2
	1	1	0	−1	−2	−3
	0	0	−1	−2	−3	−4



c Euler's method:

$$y(x+h) = y(x) + h \times \frac{\mathrm{d}y}{\mathrm{d}x}(x)$$

$$y(2) = 2, h = 0.1$$

From GDC: $y(2.5) \approx 1.89$

5 **a** Euler's method:

$$y(x+h) = y(x) + h \times \frac{\mathrm{d}y}{\mathrm{d}x}(x)$$

$$y(1) = 5, h = 0.05$$

From GDC: $y(2) \approx 5.058$

b

$$\int y^2 \ \mathrm{d}y = \int x \ \mathrm{d}x$$

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 + c$$

$$y = \sqrt[3]{1.5x^2 + 3c}$$

$$y(1) = 5 = \sqrt[3]{1.5 + 3c}$$

$$c = \frac{1}{3}(125 - 1.5) = \frac{247}{6}$$

$$y = \sqrt[3]{\frac{3x^2 + 247}{2}}$$

5 c

$$\text{Percentage error} = \frac{|\text{estimated value} - \text{actual value}|}{|\text{actual value}|} \times 100\%$$

$y(2) = 5.059$ so Percentage error $= 0.0256\%$

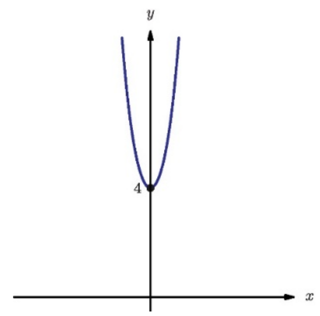
6 a

$$\int y^{-0.5}\ \mathrm{d}y = \int 4x\ \mathrm{d}x$$

$$2\sqrt{y} = 2x^2 + k$$

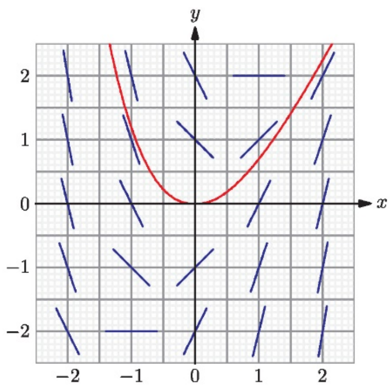
$$y = (x^2 + c)^2$$

b



7 a, b

		x = -2	-1	0	1	2
	2	-6	-4	-2	0	2
	1	-5	-3	-1	1	3
y	0	-4	-2	0	2	4
	-1	-3	-1	1	3	5
	-2	-2	0	2	4	6



8 a

$$\frac{\mathrm{d}V}{\mathrm{d}t} = k\sqrt{V}$$

When $V = 225$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 90 = 15k$$

$$k = 6$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 6\sqrt{V}$$

8 **b**

$$\int V^{-0.5}\ \mathrm{d}V = \int 6\ \mathrm{d}t$$
$$2\sqrt{V} = t + c$$
$$V = (3t + k)^2$$

c $V(0) = 225 = k^2$

$\Rightarrow k = 15$

$$V = (3t + 15)^2$$

When $V = 2500$,

$3t + 15 = 50$

$$t = \frac{35}{3} = 11.7 \text{ seconds}$$

9 **a** Eigenvalues are λ such that $\det\begin{pmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{pmatrix} = 0$

$$\lambda^2 - 6\lambda + 9 - 4 = 0$$
$$\lambda^2 - 6\lambda + 5 = 0$$
$$(\lambda - 1)(\lambda - 5) = 0$$

b $\begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$

So eigenvalue 1 corresponds to eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

and 5 to eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + B\mathrm{e}^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

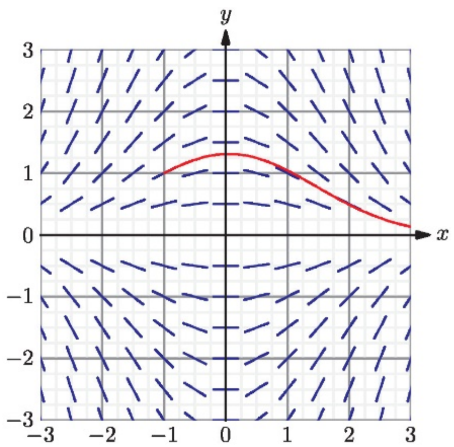
10 Euler's method:

$x(t + h) = x(t) + h \times \dot{x}(t)$
$y(t + h) = y(t) + h \times \dot{y}(t)$

From GDC: $x(0) = 0, y(0) = 2, h = 0.1$

This gives $x(0.6) \approx -13.6,\ \ y(0.6) \approx 25.4$

11 **a** Estimated curve:



Maximum value of y is approximately 1.3.

b As $x \to \infty, y \to 0$

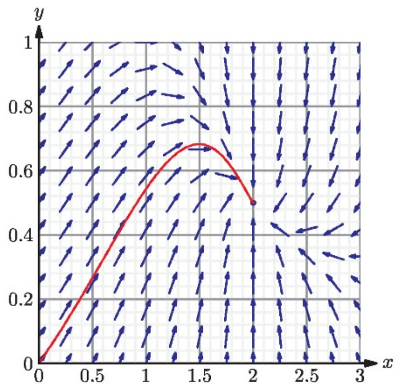
12 Estimating from the curve:

a Maximum x is 2.3.

b Maximum y is 1.7.

c Curve tends towards stable equilibrium at $(1, 1)$.

13 a Estimated curve:



Maximum y is 0.7.

b The long-term value of x is 2.

14 a Euler's method:

$$y(x+h) = y(x) + h \times \frac{dy}{dx}(x)$$

From GDC: $y(0) = 1, h = 0.1$

This gives $y(0.4) \approx 1.57$

b The slope is always positive and increasing as x increases, so the Euler approximation underestimates the actual curve, since the tangent will always run under the actual curve.

15 a

$$\int R^{-1}\ dR = \int -k\ dt$$

$$\begin{aligned} \ln|R| &= -kt + c \\ R &= Ae^{-kt} \\ R(0) &= R_0 = A \\ R &= R_0e^{-kt} \end{aligned}$$

b

$$\text{When } R = \frac{1}{2}R_0$$

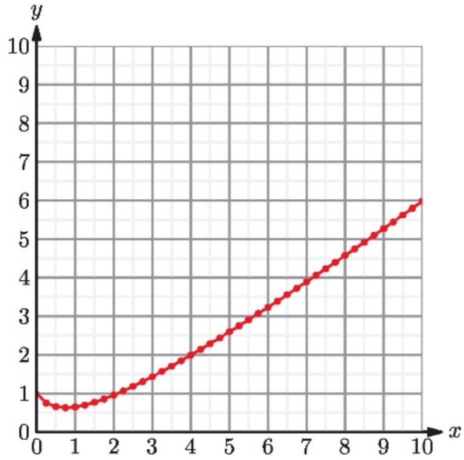
$$e^{-kt} = \frac{1}{2}$$

$$t = \frac{1}{k}\ln 2$$

c The time taken to halve the amount of the substance is independent of the amount at the start; that is, half-life is independent of the amount of the substance involved.

16 a Euler's method:

$$y(x+h) = y(x) + h \times \frac{dy}{dx}(x)$$



$y(0) = 1, h = 0.25$

b The minimum value of y is approximately 0.6.

17 a Euler's method:

$$y(x+h) = y(x) + h \times \frac{dy}{dx}(x)$$

$y(0) = -1, h = 0.1$

From GDC: $y(2) \approx -0.599$

b

$$\int e^{-y}\ dy = \int e^{-x}\ dx$$

$$-e^{-y} = -e^{-x} + c$$

$$y = -\ln(e^{-x} - c)$$

$$y(0) = -1 = -\ln(1 - c) \text{ so } c = 1 - e$$

$$y = -\ln(e^{-x} + e - 1)$$

Then $y(2) = -0.617$

Error $= 0.0178$

c To reduce error in estimate, use a smaller step length.

18

$$\int N^{-1}\ dN = \int 0.2 + 0.4\sin\left(\frac{\pi t}{6}\right)\ dt$$

$$\ln|N| = 0.2t - \frac{2.4}{\pi}\cos\left(\frac{\pi t}{6}\right) + c$$

$$N = Ae^{0.2t - \frac{2.4}{\pi}\cos\left(\frac{\pi t}{6}\right)}$$

$$N(0) = 2 = Ae^{-\frac{2.4}{\pi}}$$

$$A = 2e^{\frac{2.4}{\pi}}$$

$$N = 2e^{0.2\left(t - \frac{12}{\pi}\left(1 - \cos\left(\frac{\pi t}{6}\right)\right)\right)}$$

19 a Euler's method:

$$y(x+h) = y(x) + h \times \frac{\mathrm{d}y}{\mathrm{d}x}(x)$$
$$y(1) = 2, h = 0.1$$

From GDC: $y(1.3) \approx 3.92$

b

$$\int y^{-1}\ \mathrm{d}y = \int 3x^{-2}\ \mathrm{d}x$$
$$\ln|y| = -3x^{-1} + c$$
$$y = A\mathrm{e}^{-3x^{-1}}$$
$$y(1) = 2 = A\mathrm{e}^{-3}$$
$$y = 2\mathrm{e}^{3(1-x^{-1})}$$

c $y(1.3) = 3.997$

$$\text{Percentage error} = \frac{|\text{estimated value} - \text{actual value}|}{|\text{actual value}|} \times 100\% = 1.90\%$$

d To improve the accuracy of the approximation, reduce the step length.

20 a

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t} - \frac{\mathrm{d}C}{\mathrm{d}t} = -0.02R - 0.01C$$

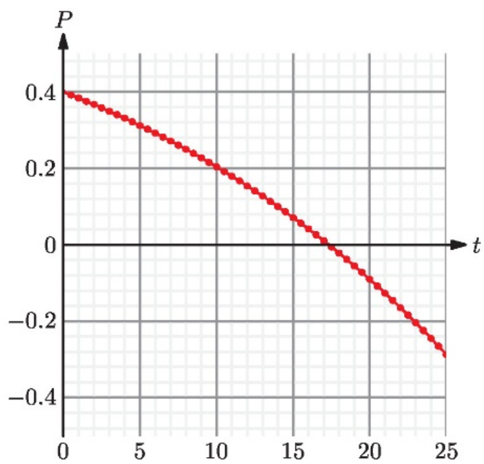
Since R and C are positive variables, it follows that $\dot{P} < 0$ and so profit is decreasing.

b Euler's method:

$$R(t+h) = R(t) + h \times \dot{R}(t)$$
$$C(t+h) = C(t) + h \times \dot{C}(t)$$
$$R(0) = 1, C(0) = 0.6, h = 0.5$$

i From GDC: $R(10) \approx \$1.31$

ii



iii Profit falls to a negative value after approximately 17 years.

21 a

$$\int m^{-1}\ \mathrm{d}m = \int -k\ \mathrm{d}t$$
$$\ln|m| = -kt + c$$
$$m = A\mathrm{e}^{-kt}$$

b The time taken to fall from $m(0) = A$ to $m(7) = 0.5A$ is the half-life.

$$\mathrm{e}^{-7k} = 0.5$$
$$k = \frac{1}{7}\ln 2 \approx 0.0990$$

c

$$m(T) = 0.1A$$
$$T = \frac{1}{k}\ln 10 \approx 23 \text{ years}$$

22 a

$$\int (y+2)^{-1}\ \mathrm{d}y = \int x - 1\ \mathrm{d}x$$
$$\ln|y+2| = \frac{1}{2}x^2 - x + c$$
$$y = A\mathrm{e}^{\frac{1}{2}x^2 - x} - 2$$

b

$$y(1) = 5 = A\mathrm{e}^{-0.5} - 2$$
$$A = 7\mathrm{e}^{0.5}$$
$$y = 7\mathrm{e}^{\frac{1}{2}(x-1)^2} - 2$$

Then $y(3) \approx 49.7$

23

$$\int \mathrm{e}^{y}\ \mathrm{d}y = \int 6\cos 2x\ \mathrm{d}x$$
$$\mathrm{e}^{y} = 3\sin 2x + c$$
$$y = \ln(3\sin 2x + c)$$

24 Euler's method:

$x(t+h) = x(t) + h \times \dot{x}(t)$
$y(t+h) = y(t) + h \times \dot{y}(t)$
$x(0) = 1, y(0) = 2, h = 0.1$
From GDC, $x(1) \approx 9.46, y(1) \approx 3.71$

25 a $\ddot{x} + 4x = \mathrm{e}^{-t}$

If $y = \dot{x}$ then $\dot{y} = \ddot{x}$

So $\dot{y} = \mathrm{e}^{-t} - 4x$

b Euler's method:

$x(t+h) = x(t) + h \times \dot{x}(t)$
$y(t+h) = y(t) + h \times \dot{y}(t)$
$x(0) = 5, y(0) = -1, h = 0.1$
From GDC, $x(0.8) \approx -0.427$

26 a Require $\frac{d\theta}{dt} = k\cos\theta$

$$\theta(0) = \frac{\pi}{3}, \frac{d\theta}{dt}(0) = 0.1$$

$$\text{So } 0.1 = \frac{k}{2}$$
$$k = 0.2$$
$$\frac{d\theta}{dt} = 0.2\cos\theta$$

b Euler's method:

$$\theta(t+h) = \theta(t) + h \times \dot{\theta}(t)$$
$$\theta(0) = \frac{\pi}{3}, \dot{\theta}(0) = -0.1, h = 0.1$$

From GDC, $\theta(2) \approx 1.213$ radians, or 69.7°

27 a $v(0) = 0, \dot{v}(0) = 0.5$

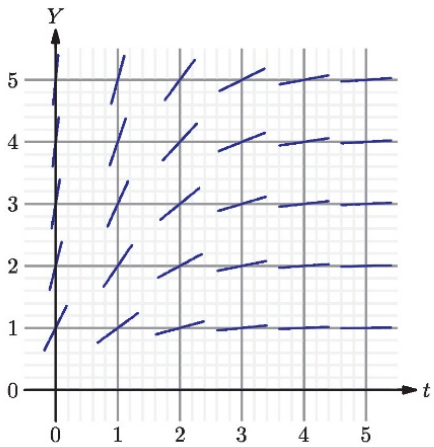
$$\text{Acceleration } a = \dot{v} = k(10 - 0.2v^2)$$
$$\text{So } 0.5 = 10k$$
$$\Rightarrow k = 0.05$$
$$\dot{v} = 0.05(10 - 0.2v^2)$$

b Euler's method:

$$v(t+h) = v(t) + h \times \dot{v}(t)$$
$$v(0) = 0, \dot{v}(0) = 0.5, h = 0.05$$

From GDC, $v(2) \approx 0.994$ m s^{-1}

28 a



b

$$\int Y^{-1}\, dY = \int 2e^{-t}\, dt$$
$$\ln|Y| = -2e^{-t} + c$$
$$Y = Ae^{-2e^{-t}}$$
$$Y(0) = 1 = Ae^{-2}$$
$$\Rightarrow A = e^2$$
$$Y = e^{2-2e^{-t}}$$

28 c When $Y = 2$

$$
\begin{aligned}
e^{2-2e^{-t}} &= 2 \\
2 - 2e^{-t} &= \ln 2 \\
e^{-t} &= 1 - \frac{1}{2}\ln 2 \\
t &= -\ln\left(1 - \frac{1}{2}\ln 2\right) \approx 0.426
\end{aligned}
$$

d $Y \to e^2 \approx 7.39$ as $t \to \infty$

The model predicts that the number of phones in the country will eventually be 739 000.

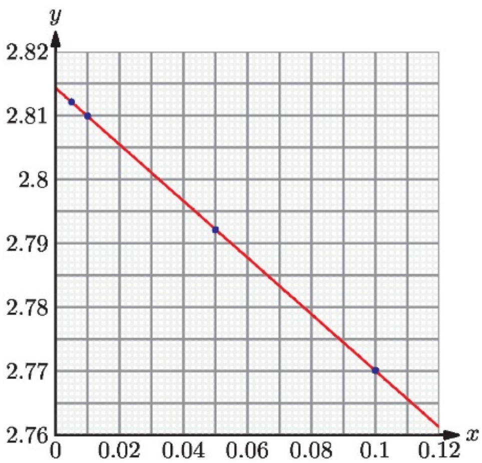
29 a Euler's method:

$$y(x+h) = y(x) + h \times \frac{dy}{dx}(x)$$

$$y(1) = 1, h = 0.1$$

From GDC, $y(2) \approx 2.7701$

b, c



d The true value of c will be found as $h \to 0$.

From the best line fit, this will be approximately 2.814.

30

Let $z = \frac{dy}{dx}$

Then $\frac{dz}{dx} = \frac{d^2y}{dx^2} = -xe^{-x^2}$

Euler's method:

$$y(x+h) = y(x) + h \times z(x)$$

$$z(x+h) = z(x) + h\left(-xe^{-x^2}\right)$$

$$y(0) = 0, z(0) = 1, h = 0.1$$

From GDC, $y(1) \approx 0.904$

31

Let $z = \dfrac{\mathrm{d}y}{\mathrm{d}x}$

Then $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x + y$

Euler's method:

$y(x+h) = y(x) + h \times z(x)$

$z(x+h) = z(x) + h\big(2x + y(x)\big)$

a $y(0) = 1, z(0) = 2, h = 0.1$

Then $y(0.1) \approx y(0) + 0.1z(0) = 1.2$

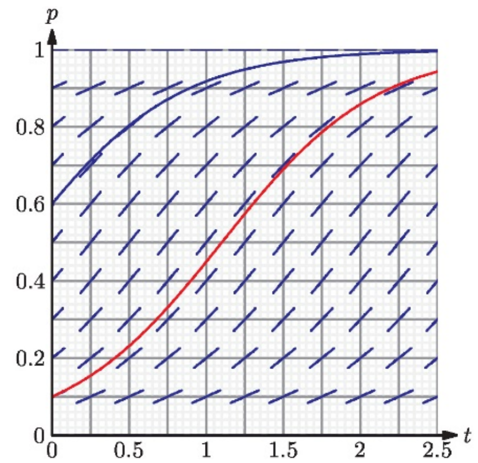
$\dfrac{\mathrm{d}y}{\mathrm{d}x}(0.1) = z(0.1) \approx z(0) + 0.1(2 \times 0 + 1) = 2.1$

b From GDC, $y(1) \approx 3.96$

32 a

$$\frac{1}{x} + \frac{1}{1-x} = \frac{1-x}{x(1-x)} + \frac{x}{x(1-x)} = \frac{1}{x(1-x)}$$

b



32 c i

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 2p(1-p)$$

$$\int \frac{1}{p(1-p)}\,\mathrm{d}p = \int 2\,\mathrm{d}t$$

Using part **a**:

$$\int \frac{1}{p} + \frac{1}{1-p}\,\mathrm{d}p = \int 2\,\mathrm{d}t$$

$$\ln|p| - \ln|1-p| = 2t + c$$

Substituting $p(0) = 0.1$: $\ln 0.1 - \ln 0.9 = c = -\ln 9$

$$\ln\left|\frac{9p}{1-p}\right| = 2t$$

$$\frac{9p}{1-p} = \mathrm{e}^{2t}$$

$$(1-p)\mathrm{e}^{2t} = 9p$$

$$p(9 + \mathrm{e}^{2t}) = \mathrm{e}^{2t}$$

$$p = \frac{\mathrm{e}^{2t}}{9 + \mathrm{e}^{2t}}$$

$$p = \frac{1}{1 + 9\mathrm{e}^{-2t}}$$

ii Rearranging:

$$1 + 9\mathrm{e}^{-2t} = \frac{1}{p}$$

$$t = -\frac{1}{2}\ln\left(\frac{1}{9}\left(\frac{1}{p} - 1\right)\right)$$

From the equation, $p = 0.5$ at $t = 1.1$

It takes about 1.1 weeks for half the people to know the rumour.

33 a Let $y = \dot{x}$

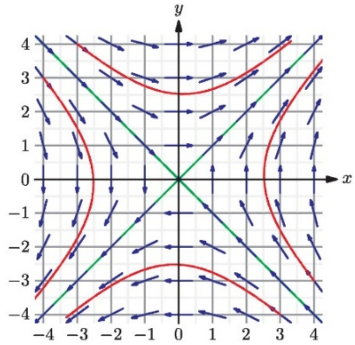
Then $\dot{y} = \ddot{x} = x$

So the system is $\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases}$

33 b i

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{x}{y}$$

The slope is 1 along $y=x$, -1 along $y=-x$, and in general is k for a point on $y=\frac{1}{k}x$.



ii From the phase plane diagram, it is clear that the origin represents a saddle point.

Trajectories approach along eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$ and are directed away along $\begin{pmatrix}1\\1\end{pmatrix}$.

c i The diagram does not show t values explicitly.
The path of a trajectory can be sketched,
but the rate at which points move along that path is not shown.

ii Euler's method:

$x(t+h)=x(t)+h\times y(t)$
$y(t+h)=y(t)+h\times x(t)$
$x(0)=0, y(0)=2, h=0.2$

From GDC, $x(2)\approx 6.08$

iii Because $\dot{x}>0$ and is increasing throughout the forward trajectory, the Euler method will consistently underestimate its growth.
The tangent at any point will pass below the curve, but the Euler approximation uses the endpoint of tangent line segments as its estimated values.

d i $\begin{pmatrix}\dot{x}\\\dot{y}\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

Eigenvalues are λ such that $\det\begin{pmatrix}-\lambda&1\\1&-\lambda\end{pmatrix}=0$

$\lambda^2-1=0$
$\lambda=\pm 1$

Then $x=Ae^t+Be^{-t}$
$x(0)=0\Rightarrow A+B=0$
$\dot{x}=Ae^t-Be^{-t}$
$\dot{x}(0)=2\Rightarrow A-B=2$
$A=1, B=-1$
$x=e^t-e^{-t}$

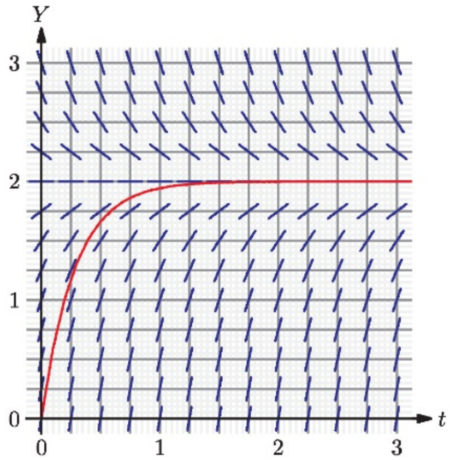
33 d ii From part **c i**, $x(2) = 7.253$

$$\text{Percentage error} = \frac{|\text{approximate value} - \text{true value}|}{|\text{true value}|} \times 100\%$$
$$= 16.1\%$$

iii The error would be expected to reduce with a smaller step size.

34 a i, ii

Y	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5	2.75
$\dot{Y}$	6	5.25	4.5	3.75	3	2.25	1.5	0.75	0	−0.75	−1.5	−2.25

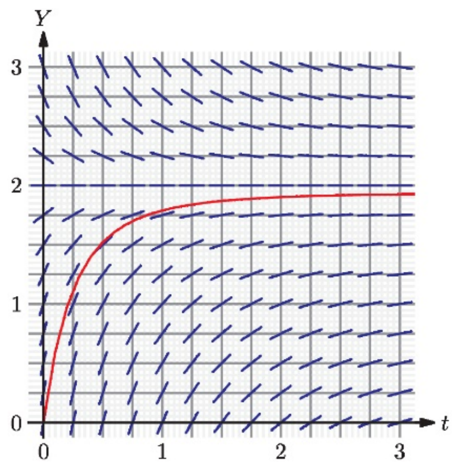


iii Eventually the model predicts 2000 plants will be infected.

b

$$\frac{dY}{dt} = 6 - 3Y = -3(Y - 2)$$
$$\int \frac{1}{Y-2}\, dY = \int -3\, dt$$
$$\ln|Y - 2| = -3t + c$$
$$Y = 2 + Ae^{-3t}$$
$$Y(0) = 0 = 2 + A \Rightarrow A = -2$$
$$Y = 2(1 - e^{-3t})$$

c i



The slope field will be the same at $t = 0$ as in the original model, and the slopes for increasing t will be increasingly horizontal at every value of Y.

34 c ii

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -3(Y-2)\mathrm{e}^{-2t}$$
$$\int \frac{1}{Y-2}\,\mathrm{d}Y = \int -3\mathrm{e}^{-2t}\,\mathrm{d}t$$
$$\ln|Y-2| = 1.5e^{-2t} + c$$
$$Y = 2 + A\mathrm{e}^{1.5\mathrm{e}^{-2t}}$$
$$Y(0) = 0 = 2 + A\mathrm{e}^{\frac{3}{2}} \Rightarrow A = -2\mathrm{e}^{-1.5}$$
$$Y = 2\left(1 - \mathrm{e}^{1.5(\mathrm{e}^{-2t}-1)}\right)$$

iii In this new model, $Y \to 2(1 - \mathrm{e}^{-1.5}) \approx 1.55$.

The model predicts 1550 plants will eventually be infected.

35 a

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -0.2S + 0.4I$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0.2S - 0.4I$$

b i Let $P = S + I$

Then $\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}I}{\mathrm{d}t} = 0$

So P is constant.

ii

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\mathrm{d}I}{\mathrm{d}t}$$
$$\frac{\mathrm{d}S}{\mathrm{d}I} = \frac{\mathrm{d}S}{\mathrm{d}t} \div \frac{\mathrm{d}I}{\mathrm{d}t} = -1$$

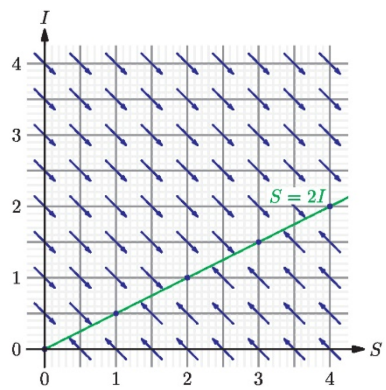
c

If $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$

then $0.4I = 0.2S$

$S = 2I$

d Slope is -1 except on line S$= 2I$, which is all at equilibrium.
All movement is towards the line.



35 e i Since $S + I$ is constant and $S(0) + I(0) = 15$,

transform the differential equation using $I = 15 - S$.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -0.2S + 0.4I = 6 - 0.6S$$

$$\int \frac{1}{S - 10}\,\mathrm{d}S = \int -0.6\ \mathrm{d}t$$

$$\ln|S - 10| = -0.6t + c$$

$$S = 10 + Ae^{-0.6t}$$

$$S(0) = 14 = 10 + A \Rightarrow A = 4$$

$$S = 10 + 4e^{-0.6t}$$

ii As $t \to \infty, S -> 10$

In the long term, there are 5 million infected individuals, 10 million susceptibles.

f The population in the model is constant, suggesting that net birth, death and migration effects cancel out.

There is no third class of 'resistant' or 'immune' individuals; all infected revert to susceptible. In real life, for most disease processes, a proportion of the recovered individuals might be expected to be less susceptible to reinfection.

There is no interaction effect. In many disease-spread models, increase in infected individuals is proportional to the expected number of meetings of susceptible and infected individuals; in a well-mixed population, this is often shown as SI, the product of the two population sizes.

Applications and interpretation: Practice Paper 1

Solutions

1 a

$$\mathrm{P}(X > 25 | X > 11) = \frac{\mathrm{P}(X > 25 \cap X > 11)}{\mathrm{P}(X > 11)} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{3}$$

b Let A be the event that a leaf length lies between 11 cm and 25 cm

$$\mathrm{P}(A) = \frac{1}{2}$$

Let Y be the number of leaves with length between 11 cm and 25 cm in a random sample of 5.

$$Y \sim \mathrm{B}\left(5, \frac{1}{2}\right)$$

$$\mathrm{P}(Y = 2) = \frac{5}{16}$$

2 a $\log_{10}(100ab^2) = \log_{10}(100) + \log_{10} a + 2\log_{10} b = 2 + x + 2y$

b

$$\log_b a = \frac{\log a}{\log b} = \frac{x}{y}$$

3 a From the GDC, maximum for $0 \leq x \leq 12$ is $y(4) = \$464$.

b From the GDC, maximum for $0 \leq x \leq 18$ is $y(18) = \$632$.

c In the long term, the model predicts x increasing without limit at an increasing gradient; profits increasing faster and faster with no limit is unrealistic.

4 a $u_5 = 5 = a + 4d$ (1)

$u_{10} = -15 = a + 9d$ (2)

$(2) - (1)$: $5d = -20 \Rightarrow d = -4$

(1): $a = 5 - 4d = 21$

The first term is $a = 21$, the common difference is $d = -4$.

b $S_n = \frac{n}{2}\left(2a + d(n-1)\right) = n$

$2a + d(n-1) = 2$

$42 - 4(n-1) = 2$

$n - 1 = 10$

$n = 11$

5 a $\mathrm{f}(0) = 16$ and $\mathrm{f}(t) \to 0$ as $t \to \infty$ with the function always decreasing, so the range is $0 < \mathrm{f}(t) \leq 16$.

b Let $y = \mathrm{f}(t) = 16\mathrm{e}^{-3t}$

$$\frac{y}{16} = \mathrm{e}^{-3t}$$

$$t = -\frac{1}{3}\ln\left(\frac{y}{16}\right) = \frac{1}{3}\ln\left(\frac{16}{y}\right) = \mathrm{g}(y)$$

Changing variables:

$$\mathrm{g}(t) = \frac{1}{3}\ln\left(\frac{16}{t}\right)$$

c $\mathrm{g}(8) = \frac{1}{3}\ln 2 \approx 0.231$ years

6 a

$$\frac{z}{w^*} = \frac{1+2\mathrm{i}}{2+\mathrm{i}} = \frac{(1+2\mathrm{i})(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})} = \frac{4+3\mathrm{i}}{5}$$

b $pz + qw = p + 2q + \mathrm{i}(2p - q) = \mathrm{i}$

Comparing real and imaginary parts:

$$\begin{cases} p + 2q = 0 & (1) \\ 2p - q = 1 & (2) \end{cases}$$

$$2(1) - (2)\text{: } 5q = -1 \Rightarrow q = -\frac{1}{5}$$

$$(1)\text{: } p = -2q = \frac{2}{5}$$

7 a

$$\frac{1}{3x\sqrt{x}} = \frac{1}{3}x^{-\frac{3}{2}}$$

b

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3x\sqrt{x}}\right) = -\frac{1}{2}x^{-\frac{5}{2}}$$

c

$$\int \frac{1}{3x\sqrt{x}}\,\mathrm{d}x = c - \frac{2}{3}x^{-\frac{1}{2}}$$

8 a $u_{n+1} = 1.04u_n + 100$

b If $a = 100$ and $r = 1.04$ then

$u_{18} = a$

$u_{19} = ar + a = a(1 + r)$

$u_{20} = ra(1 + r) + a = a(1 + r + r^2)$

It is clear that u_n is the geometric series, summing terms of a geometric sequence.

$$u_{18+n} = a\sum_{k=0}^{n} r^k = a\frac{(r^{n+1} - 1)}{r - 1}$$

When $u_{18+n} > 2000$:

$$100\left(\frac{1.04^{n+1} - 1}{1.04 - 1}\right) > 2000$$

$$1.04^{n+1} - 1 > 0.8$$

$$1.04^{n+1} > 1.8$$

$$n + 1 > \frac{\ln(1.8)}{\ln(1.04)} = 14.9$$

Least such n is $n = 14$

Then $u_{18+14} = u_{32}$ is the first term exceeding 2000.

On his 32nd birthday, Morgan will have more than \$2000 in his account for the first time.

c $u_{32} = \frac{100(1.04^{15}-1)}{0.04} = 2002.36$

Of that amount, \$1500 is from the 15 sums of \$100.

The interest is therefore 502.36 which represents $\frac{502.36}{2002.36} \times 100\% = 25.1\%$ of the balance.

9

$$y = 2^x = \mathrm{e}^{x\ln 2} \text{ so } x = \frac{\ln y}{\ln 2}$$

$$V = \pi\int_2^4 x^2\ \mathrm{d}y = \pi\int_2^4 \left(\frac{\ln y}{\ln 2}\right)^2\ \mathrm{d}y = 15.7\ \mathrm{cm}^2\ (\mathrm{GDC})$$

10 a

$$x = \frac{-1 \pm \sqrt{1^2 - 4(3)(4)}}{2(3)} = \frac{-1 \pm \mathrm{i}\sqrt{47}}{6}$$

b The angle between the two conjugates equals double the angle from the horizontal.

$$2\arctan\left(\frac{\sqrt{47}}{1}\right) = 2.85$$

11 a Room temperature would be shown by the horizontal asymptote, 19 °C

b $T(0) = A + 19 = 82 \Rightarrow A = 63$

c $T(8) = 45 = 19 + 63e^{-8k}$

$k = -\frac{1}{8}\ln\left(\frac{26}{63}\right) \approx 0.11$

d $T(16) = 19 + 63\left(\frac{26}{63}\right)^2 \approx 30\,°\text{C}$

12 a $(8, 2)$ is within region around station B, so the temperature estimate is 18 °C.

b The station will be at one of the two region vertices; the lower of the two vertices is further from A, B and C than the upper is from A, C and D.

The vertex is at approximately $(5.8, 2.2)$.

13a Trapezoidal rule approximation:

$x_0 = 0, n = 4, h = \frac{\sqrt{\pi}}{4}$

x	y	$\times$	$=$
$x_0 = 0$	0	$\times 1$	0
$x_1 = \frac{\sqrt{\pi}}{4}$	0.519	$\times 2$	1.03737
$x_2 = \frac{\sqrt{\pi}}{2}$	3.760	$\times 2$	7.51988
$x_3 = \frac{3\sqrt{\pi}}{4}$	7.823	$\times 2$	15.64557
$x_4 = \sqrt{\pi}$	0	$\times 1$	0
	TOTAL		24.20282
	TOTAL $\times \frac{h}{2}$		5.36230

Approximate area: 5.3623

b

$$\int_0^{\sqrt{\pi}} 6x\sin(x^2)\ \mathrm{d}x = 6 \text{ (GDC)}$$

c

$$\begin{aligned}\text{Percentage error} &= \frac{|\text{true value} - \text{approximate value}|}{\text{true value}} \times 100\% \\ &= \frac{|6 - 5.3623|}{6} \\ &= 10.6\%\end{aligned}$$

14 $v = \frac{\mathrm{d}s}{\mathrm{d}t} = \ln(t+1) + \frac{t}{t+1} = \ln(t+1) + 1 - \frac{1}{t+1}$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{t+1} + \frac{1}{(t+1)^2} = \frac{t+2}{(t+1)^2}$$

15 a $y(1.2) = 6.14 = A - 1.271B - 0.165C$

$y(3.5) = 5.70 = A - 2.014B - 0.0.0052C$

$y(6.3) = 2.95 = A - 3.525B - 0.00008C$

Solving simultaneous equations on the GDC:

$A = 9.4, B = 1.8, C = 5.8$

b From the GDC, with these values, the maximum x associated with a positive value for the population is just above 8. The highest water temperature that can support this species of fish is 8 °C.

16 a $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 6 \\ 1 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 8 \\ 4 \\ h-5 \end{pmatrix}$

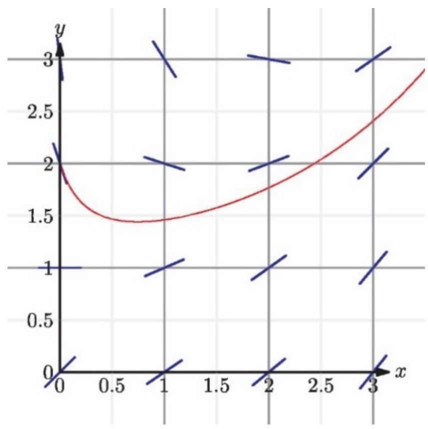
If $\overrightarrow{AB} \perp \overrightarrow{AC}$, then $\overrightarrow{AC} \cdot \overrightarrow{AB} = 0$

$\overrightarrow{AC} \cdot \overrightarrow{AB} = -32 + 24 + h - 5$

$h = 13$

b Area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|\begin{pmatrix} -4 \\ 6 \\ 1 \end{pmatrix} \times \begin{pmatrix} 8 \\ 4 \\ 8 \end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix} 44 \\ 40 \\ -64 \end{pmatrix}\right| = 43.7$

17 a



b

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x - y^2}{(1+x)^2} = \mathrm{f}(x, y)$$

$y(x+h) = y(x) + h \times \mathrm{f}\big(x, y(x)\big)$

From the GDC: $y(0.3) \approx 1.47$

18 a

$$\frac{dy}{dt} = -0.06(t-1)(y-5)$$

Separation of variables:

$$\int \frac{1}{y-5}\, dy = \int -0.06(t-1)\ dt$$

$$\ln|y-5| = c - 0.03(t-1)^2$$

$$y - 5 = Ae^{-0.03(t-1)^2}$$

$$y_{GS} = 5 + Ae^{-0.03(t-1)^2}$$

$$y(1) = 15 = 5 + A \Rightarrow A = 10$$

$$y_{PS} = 5 + 10e^{-0.03(t-1)^2}$$

b As $t \to \infty$, $y \to 5$ so in the long term, the bird population decreases to 500.

19 $V_1 = 15\sin(30t)$

$V_2 = 20\sin(30t+5)$

Let $z_1 = 15 + 20e^{5i} = 15 + 20\cos 5 + 20i\sin 5 \approx 20.67 - 19.18i$

Let $z_2 = e^{30ti} = \cos 30t + i\sin 30t$

$|z_1| = \sqrt{20.76^2 + 19.18^2} \approx 28.2$

$\text{Arg}(z_1) = \arctan\left(-\frac{19.18}{20.67}\right) \approx 0.748$

So $z_1 \approx 28.2e^{-0.748i}$

$$\begin{aligned}\text{Im}(z_1 z_2) &= \text{Im}\left(\left(15 + 20e^{5i}\right)e^{30ti}\right)\\ &= \text{Im}\left(15e^{30ti} + 20e^{(30t+5)i}\right)\\ &= 15\sin 30t + 20\sin(30t+5)\\ &= V_1 + V_2\end{aligned}$$

$V_1 + V_2 = \text{Im}(z_1 z_2) = \text{Im}\left(28.2e^{(30t-0.748)i}\right) = 28.2\sin(30t - 0.748)$

Applications and interpretation: Practice Paper 2

Solutions

1 **a** Cylinder surface area: $2\pi rh = 84\pi\ \text{m}^2$

Dome surface area: $\frac{2}{3}\pi r^3 = 18\pi\ \text{m}^2$

Total surface area: $102\pi = 320\ \text{m}^2$

b $\text{AB} = \sqrt{14^2 + 25^2} = 28.7\ \text{m}$

c $\text{A}\hat{\text{B}}\text{C} = 180° - \tan^{-1}\left(\frac{14}{25}\right) \approx 151°$

d So $\text{A}\hat{\text{C}}\text{B} = 162 - 151 = 11.2°$

Sine rule:

$$\text{BC} = \frac{\text{AB}}{\sin \text{A}\hat{\text{C}}\text{B}} \sin B\hat{A}C = \frac{28.7 \sin 18°}{\sin 11.2°} = 45.4\ \text{m}$$

e The area of water is the area between two 140° arcs, the inner having radius 25 m and the outer having radius $25 + BC = 70.4$ m

$$\text{Area} = \frac{140}{360}\pi(70.4^2 - 25^2) = 5290\ \text{m}^2$$

2 **a** Let T be the number of minutes taken by Jomer Tree students. $T \sim \text{N}(\mu, \sigma^2)$

$$\text{P}(T < 10) = 0.3 \Rightarrow \frac{10 - \mu}{\sigma} = z_{0.3} = -0.524$$

$$\text{P}(T < 12) = 0.6 \Rightarrow \frac{12 - \mu}{\sigma} = z_{0.6} = 0.253$$

$$\begin{cases} 10 - \mu = -0.524\sigma & (1) \\ 12 - \mu = 0.253\sigma & (2) \end{cases}$$

$(2) - (1): 2 = 0.778\sigma \Rightarrow \sigma = 2.57$ minutes

$(1): \mu = 10 + 0.524\sigma = 11.3$ minutes

b

Time (minutes) t	$t < 10$	$10 < t < 12$	$t > 12$
Probability	0.3	0.3	0.4
Score x	5	3	0

$\text{E}(X) = \sum_x x\,\text{P}(X = x) = 2.4$

Then the expected score for 50 students is $50 \times 2.4 = 120$

c

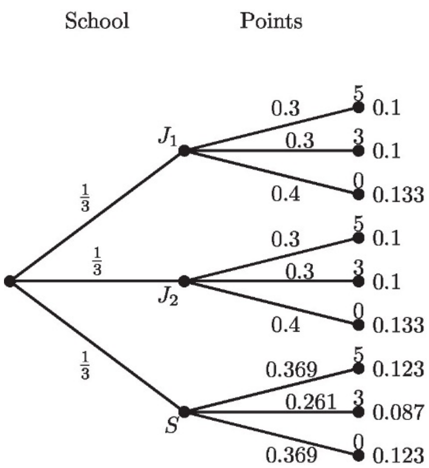
		X_1		
		5	**3**	**0**
X_2	**5**	0.09	0.09	0.12
	3	0.09	0.09	0.12
	0	0.12	0.12	0.16

$\mathrm{P}(X_1 + X_2 < 6) = 0.64$

d Let R be the time in minutes of a student from St Atistics. $R \sim \mathrm{N}(11, 3^2)$, and Y be the associated score.

$\mathrm{P}(Y = 0) = \mathrm{P}(R > 12) = 0.369$

$\mathrm{P}(Y = 5) = \mathrm{P}(R < 10) = 0.369$



$$\begin{aligned}\mathrm{P}(\text{student is from Jomer Tree} \mid \text{student scores } 0) &= \frac{\mathrm{P}(\text{student is from Jomer Tree and scores } 0)}{\mathrm{P}(\text{student scores } 0)} \\ &= \frac{0.267}{0.390} \\ &= 0.684\end{aligned}$$

The times for students from St Atistics are thought to be normally distributed with mean 11 minutes and standard deviation 3 minutes. St Atistics enters one student in the race.

e Jomer Tree wins in the following score scenarios, where the scores for the Jomer Tree students are listed first, then the St Atistics student:

$5, 5, \text{any}: 0.3 \times 0.3 \times 1 = 0.09$

$5, 3, \text{any}: 0.3 \times 0.3 \times 1 = 0.09$

$3, 5, \text{any}: 0.3 \times 0.3 \times 1 = 0.09$

$3, 3, \text{any}: 0.3 \times 0.3 \times 1 = 0.09$

$5, 0, 3: 0.3 \times 0.4 \times 0.261 = 0.0313$

$5, 0, 0: 0.3 \times 0.4 \times 0.369 = 0.0443$

$0, 5, 3: 0.4 \times 0.3 \times 0.261 = 0.0313$

$0, 5, 0: 0.4 \times 0.3 \times 0.369 = 0.0443$

$3, 0, 0: 0.3 \times 0.4 \times 0.369 = 0.0443$

$0, 3, 0: 0.4 \times 0.3 \times 0.369 = 0.0443$

Total probability is 0.6

3 **a** If the states are ordered as {donate, not donate}

$$\mathbf{T} = \begin{pmatrix} 0.72 & 0.16 \\ 0.28 & 0.84 \end{pmatrix}$$

b

$$\mathbf{T}^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.300 \\ 0.700 \end{pmatrix}$$

The probability of a student donating three years after graduating is 0.300.

c Eigenvalues are λ such that $\det(\mathbf{T} - \lambda\mathbf{I}) = 0$.

For a transition matrix, one eigenvalue is 1 and since the trace equals the sum of eigenvalues, the other is clearly 0.56.

$\underline{\lambda_1 = 1:}$

$$\begin{cases} 0.72x + 0.16y = x \\ 0.28x + 0.84y = y \end{cases}$$

$4y = 7x$

Eigenvector $v_1 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$\underline{\lambda_2 = 0.56:}$

$$\begin{cases} 0.72x + 0.16y = 0.56x \\ 0.28x + 0.84y = 0.56y \end{cases}$$

$y = -x$

Eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

d $\mathbf{P} = \begin{pmatrix} 4 & 1 \\ 7 & -1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.56 \end{pmatrix}$

e $\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$

$$\begin{aligned} \mathbf{T}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 4 & 1 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.56 \end{pmatrix}^n \begin{pmatrix} 4 & 1 \\ 7 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.56^n \end{pmatrix} \frac{1}{11} \begin{pmatrix} 1 & 1 \\ 7 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{11} \begin{pmatrix} 4 & 1 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 7 \times 0.56^n & -4 \times 0.56^n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{11} \begin{pmatrix} 4 + 7 \times 0.56^n & 4(1 - 0.56^n) \\ 7(1 - 0.56^n) & 7 + 4 \times 0.56^n \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{11} \begin{pmatrix} 4(1 - 0.56^n) \\ 7 + 4 \times 0.56^n \end{pmatrix} \end{aligned}$$

Probability of donating in year n is $\dfrac{4}{11}(1 - 0.56^n)$

f As $n \to \infty$, this probability tends to $\frac{4}{11}$.

4 a Breakdowns occur at a constant rate and independently of each other.

b Let W_n be the number of breakdowns in an n-month period.

$W_n \sim \mathrm{Po}(0.8n)$

i $\mathrm{P}(W_1 = 0) = \mathrm{e}^{-0.8} = 0.449$

ii $\mathrm{P}(W_1 \leq 2) = 0.953$ (GDC)

iii $\mathrm{P}(W_3 \geq 6) = 0.0357$

c Let V be the number of 3-month periods in which the copier breaks down at least 6 times.

$V \sim \mathrm{B}(4, 0.0357)$

$\mathrm{P}(V \geq 2) = 0.007\,28$

d $X = W_6 \sim \mathrm{Po}(4.8)$

$\mathrm{E}(X) = 4.8 = \mathrm{Var}(X)$

e $X \sim \mathrm{Po}(\lambda)$

$\mathrm{H}_0: \lambda = 4.8;$

$\mathrm{H}_1: \lambda > 4.8$

$\mathrm{P}(X \geq 9|\mathrm{H}_0) = 0.0558 > 0.05$

The data is consistent with the null hypothesis at 5% significance level. Do not reject H_0

f $\mathrm{P}(\text{Type I error}) = \mathrm{P}(\text{Reject } \mathrm{H}_0 \text{ when it is true}) = 0.0558$

g $\mathrm{P}(X \geq 10|\mathrm{H}_0) = 0.0251 < 0.05$ so he would reject the null hypothesis if there were at least 10 failures in 6 months. 10 is the critical value of his test.

$$\begin{aligned}\mathrm{P}(\text{Type I error}) &= \mathrm{P}(\text{Fail to reject } \mathrm{H}_0 \text{ when it is false})\\ &= \mathrm{P}(X < 10|\lambda = 6.6)\\ &= 0.869\end{aligned}$$

Comment: Using the current data, the probability of a type II error is, strictly speaking, 1; we have already established that he has failed to reject H_0 with the 9 breakdowns observed, but we know the null hypothesis to be false.

5 a $\mathbf{v_A} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r_A} = \begin{pmatrix}1\\2\\2\end{pmatrix}$

$|\mathbf{v_A}| = \sqrt{1 + 2^2 + 2^2} = 3$ km per minute

b $\mathbf{r_A}(t_1) = \mathbf{r_B}(t_2)$

$$\begin{cases}1 + t_1 = -5 + 1.5t_2 (1)\\ 4 + 2t_1 = 1 + 1.5t_2 (2)\\ 4 + 2t_1 = 22 - 2t_2 (3)\end{cases}$$

$(2) - (1): 3 + t_1 = 6 \Rightarrow t_1 = 3, t_2 = 6$

This solution is consistent for all three equations.

Both planes pass through $(4, 10, 10)$; plane A passes through at $t = 3$, plane B at $t = 6$.

c

$$|\mathbf{r_A} - \mathbf{r_B}| = \left|\begin{pmatrix} 6-0.5t \\ 3+0.5t \\ -18+4t \end{pmatrix}\right|$$
$$= \sqrt{(6-0.5t)^2 + (3+0.5t)^2 + (-18+4t)^2}$$

d Let $d = |\mathbf{r_A} - \mathbf{r_B}|$

From the GDC, minimum d is 6.45 km at $t = 4.45$

The controller will not need to issue an alert.

6 a Let $y = \dot{x}$, so $\ddot{x} = \dot{y}$

The second order differential equation can then be given as

$$\begin{cases} \dot{x} = y \\ \dot{y} = 3x - 2y \end{cases}$$

b i Euler's method:

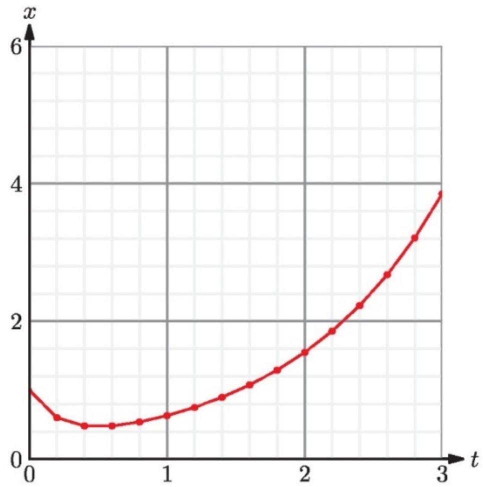
$x(t+h) \approx x(t) + h\dot{x}(t)$

$y(t+h) \approx y(t) + h\dot{y}(t)$

Using $x(0) = 1, y(0) = -2, h = 0.2$

From the GDC, $x(2) \approx 1.55$

ii



c $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$

Eigenvalues of $\mathbf{M}$ are given by $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$

$-\lambda(-2-\lambda) - 3 = 0$

$\lambda^2 + 2\lambda - 3 = 0$

$(\lambda - 1)(\lambda + 3) = 0$

$\lambda = 1 \text{ or } -3$

General solution is $x_{GS} = Ae^t + Be^{-3t}$ so $\dot{x}_{GS} = Ae^t - 3Be^{-3t}$.

$x(0) = 1 = A + B(1)$

$\dot{x}(0) = -2 = A - 3B(2)$

$(1) - (2): 4B = 3$

$B = \frac{3}{4}, A = \frac{1}{4}$

$x_{PS} = \frac{1}{4}(\mathrm{e}^t + 3\mathrm{e}^{-3t})$

This will be an unstable system, since one eigenvalue is positive and one is negative.

d

$$\text{Percentage error} = \left|\frac{\text{Approximate value} - \text{true value}}{\text{true value}}\right| \times 100\%$$
$$= 16.3\%$$

7 **a** Side length of S_1 is 2.

Side length of S_2 is $\sqrt{2}$, so the enlargement has scale factor $\frac{1}{\sqrt{2}}$.

Enlargement $\mathbf{R} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The angle of rotation is 45° anticlockwise. $\mathbf{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\mathbf{M} = \mathbf{RS} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

c T_1 has area $\frac{1}{2}$.

d $\det \mathbf{M} = \left(\frac{1}{2}\right)^2 (1 + 1) = \frac{1}{2}$

Therefore each successive triangle has half the area of its predecessor.

Area of T_2 is $\frac{1}{4}$.

e Area of the fractal is the sum of a geometric series with $r = \frac{1}{2}$ and $a = \frac{1}{2}$.

Area $= \frac{a}{1 - r} = 1$.

Worked solutions

Applications and interpretation: Practice Paper 3

Solutions

1 a The study is focused on differences due to being an older or younger sibling, so it is important to eliminate other factors which would be expected to affect speech development, such as age.

b i Let μ_1 be the true mean score for eldest children and μ_2 be the true mean score for non-eldest children.

$\mathrm{H}_0: \mu_1 = \mu_2$

$\mathrm{H}_1: \mu_1 \neq \mu_2$

ii Two sample t-test. $t = 0.766, p = 0.465 > 0.05$

Do not reject H_0; there is insufficient evidence of a difference between the groups.

iii This conclusion is overstated – there was:

- no *significant* difference (there was a difference but it was not significant, given the small number in the cohort)
- the experiment measured vocabulary as a proxy for speech development but did not actually assess speech development
- the experiment only looked at three year old children, and the result (whether significant or not) cannot be extrapolated to apply to all children.

c i Content validity would be a matter of informed opinion; consulting one (or preferably several, independent) experts in linguistic development would be appropriate

ii Using the vocabulary as a proxy for speech development is valid.

If the results of such an assessment correlate with other accepted tests.

If the results of such an assessment correlate with future measures of speech development in children.

d i

Score	$[0, 2[$	$[2, 4[$	$[4, 6[$	$[6, 8[$	$[8, 9[$	$[9, 10[$
Midpoint v	1	3	5	7	8.5	9.5
Frequency	4	20	35	40	15	5

From the GDC: $\bar{v} = 5.83,\ s_{n-1}^2 = 4.29$

ii

Score	$]-\infty, 2[$	$[2, 4[$	$[4, 6[$	$[6, 8[$	$[8, 9[$	$[9, \infty[$	ν	χ^2
Frequency	4	20	35	40	15	5		
$\mathbf{N}(\mu, \sigma^2)$	8.0	28.8	45.6	28.8	5.2	2.7		

The expected value in the final cell is less than 5 so the last two cells should be merged:

Score	$]-\infty, 4[$	$[4, 6[$	$[6, 8[$	$[8, 9[$	$[9, \infty[$	ν	χ^2
Frequency	24	35	40	15	5		
$\mathbf{N}(\mu, \sigma^2)$	22.4	40.9	38.1	10.1	7.5	4	29.7

No parameters have been estimated from the data so degrees of freedom = $5 - 1 = 4$.

$\mathrm{P}(\chi_4^2 > 29.7) = 5 \times 10^{-6} \ll 0.05$

The data is not consistent with the given $\mathrm{N}(5, 2^2)$.

iii

Score	$]-\infty, 2[$	$[2, 4[$	$[4, 6[$	$[6, 8[$	$[8, 9[$	$[9, \infty[$	ν	χ^2
Frequency	4	20	35	40	15	5		
$\mathbf{N}(\mu, \sigma^2)$	3.8	18.6	40.9	38.1	10.1	7.5		

The expected value in the first cell is less than 5 so the first two cells should be merged:

Score	$]-\infty, 4[$	$[4, 6[$	$[6, 8[$	$[8, 9[$	$[9, \infty[$	ν	χ^2
Frequency	24	35	40	15	5		
$\mathbf{N}(\mu, \sigma^2)$	22.4	40.9	38.1	10.1	7.5	2	4.33

Two parameters have been estimated from the data so degrees of freedom = $5 - 3 = 2$.

$\mathrm{P}(\chi_2^2 > 4.33) = 0.115 > 0.05$

The data is consistent with a normal distribution.

iv The chi squared test supports the necessary condition that the data be normal, but it is not established that the two groups would have the same variance, which is assumed in the pooled two sample t-test.

e i The change in mean may reflect expected development with age or with practice at the test, but does not indicate that the test is unreliable.

ii $r_s = 0.885 > 0.649$, so there is evidence of correlation, suggesting that the test is reliable.

Worked solutions

2 a i

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

ii

$$\mathbf{A}^5 = \begin{pmatrix} 60 & 69 & 55 & 89 & 66 & 78 \\ 69 & 42 & 73 & 63 & 55 & 88 \\ 55 & 73 & 42 & 88 & 69 & 63 \\ 89 & 63 & 88 & 88 & 78 & 108 \\ 66 & 55 & 69 & 78 & 60 & 89 \\ 78 & 88 & 63 & 108 & 89 & 88 \end{pmatrix}$$

There are 73 paths from B to C using exactly five roads.

b i It is not possible; some of the vertices (A, B, C, E) have odd degrees.

ii It will be necessary to repeat some roads, duplicating travel between pairs of the odd vertices.

The minimal pairing is $AB + CE$ (12).

The minimum route is therefore $63 + 12 = 75$.

This can be accomplished, for example, by $DABAFBCECDEFD$.

iii All vertices now have even degrees, if every road is duplicated, so this is possible, with a route length 126.

c i A to C: $ABC = 14, ADC = 12, AFEC = 17$. Shortest route is 12, so $a = 12$.

A to E: $ADE = 9, AFE = 11, ABCE = 20$. Shortest route is 9, so $b = 9$.

ii $F - A - D - E - C - B - F$ for a journey length of $4 + 4 + 5 + 6 + 8 + 10 = 37$

iii The two shortest connections from A are AD, AF at a total 8.

Minimum spanning tree for $BCDEF$ using Prim's algorithm: DE, EF, CE, BC: 24

Lower bound for T is $26 + 8 = 32$

iv <u>Deleting B</u>: Shortest connections to B are $AB + BC = 14$

MST for $ACDEF$: AD, AF, DE, CE: 19

Lower bound 33

<u>Deleting C</u>: Shortest connections to C are $CD + CE = 14$

MST for $ABDEF$: AD, AF, DE, AB: 19

Lower bound 33

<u>Deleting D</u>: Shortest connections to D are $AD + DE$ or $DF = 9$

MST for $ABCEF$: AF, AB, CE, BC: 23

Lower bound 32

<u>Deleting E</u>: Shortest connections to E are $CD + CE = 11$

MST for $ABCDF$: AD, AF, AB, CD or AC: 23

Lower bound 34

<u>Deleting F:</u> Shortest connections to F are $AF + DF = 9$

MST for $ABCDE$: AD, DE, CE, AB: 21

Lower bound 30

The strongest inequality by this method is $34 \leq T \leq 37$.

d Removing travel from A to D and from D to F:

Assuming an equal probability of heading along any of the available roads from each junction, the transition matrix is

$$\mathbf{T} = \frac{1}{12}\begin{pmatrix} 0 & 4 & 0 & 4 & 0 & 3 \\ 6 & 0 & 4 & 0 & 0 & 3 \\ 0 & 4 & 0 & 4 & 4 & 0 \\ 0 & 0 & 4 & 0 & 4 & 3 \\ 0 & 0 & 4 & 4 & 0 & 3 \\ 6 & 4 & 0 & 0 & 4 & 0 \end{pmatrix}$$

$$\mathbf{T}^{50}\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ 1-a-b-c-d-e \end{pmatrix} = \begin{pmatrix} 0.159 \\ 0.182 \\ 0.162 \\ 0.153 \\ 0.153 \\ 0.191 \end{pmatrix} = \frac{1}{314}\begin{pmatrix} 50 \\ 57 \\ 51 \\ 48 \\ 48 \\ 60 \end{pmatrix}$$

The model predicts that the most frequently visited junction is F, so the petrol station should be built there.